

Original Prob 14 of 1st edition below.

Problem 14. Use Noether's theorem for scalars and the transformation $x^i \rightarrow x^i + \alpha^i$ to show that three-momentum k_i is conserved. Then, show the same result via commutation of the three-momentum operator of Chap. 3 (which can be found in Wholeness Chart 5-4 at the end of Chap. 5) with the Hamiltonian.

Prob 14, Correction version of 2nd edition.

Problem 14. Show that the total (not density) 3-momentum k^i for free scalars is conserved. Use our knowledge that the conjugate momentum for x^i is k_i , the total (not density) 3-momentum (expressed in covariant components), and it is conserved if L is symmetric (invariant) under the coordinate translation transformation $x^i \rightarrow x'^i = x^i + \alpha^i$, where α^i is a constant 3D vector. Then, show the same result via commutation of the three-momentum operator of Chap. 3 (see Wholeness Chart 5-4, pg. 158) with the Hamiltonian. (Solution is posted on book website. See pg.xvi, opposite pg. 1.)

Ans. (first part):

The Lagrangian density is $\mathcal{L}_0^0 = \phi_{,\mu}^\dagger \phi^{,\mu} - \mu^2 \phi^\dagger \phi$. We must integrate this over all volume to get the total Lagrangian L . $L = \int \mathcal{L}_0^0 dV$. If k_i is conserved, then of course, so is k^i . So, we need to show L is invariant under $x^i \rightarrow x'^i = x^i + \alpha^i$.

The 1st term in \mathcal{L}_0^0 , $\phi_{,\mu}^\dagger \phi^{,\mu}$

$$\begin{aligned} \phi &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(a(\mathbf{k}) e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k}) e^{ik_\mu x^\mu} \right) & \phi^\dagger &= \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} \left(b(\mathbf{k}) e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k}) e^{ik_\mu x^\mu} \right) \\ \phi_{,\mu} &= \sum_{\mathbf{k}} \frac{ik_\mu}{\sqrt{2V\omega_{\mathbf{k}}}} \left(-a(\mathbf{k}) e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k}) e^{ik_\mu x^\mu} \right) \\ \phi^{,\mu} &= \sum_{\mathbf{k}} \frac{ik^\mu}{\sqrt{2V\omega_{\mathbf{k}}}} \left(-a(\mathbf{k}) e^{-ik_\mu x^\mu} + b^\dagger(\mathbf{k}) e^{ik_\mu x^\mu} \right) & \phi_{,\mu}^\dagger &= \sum_{\mathbf{k}} \frac{ik_\mu}{\sqrt{2V\omega_{\mathbf{k}}}} \left(-b(\mathbf{k}) e^{-ik_\mu x^\mu} + a^\dagger(\mathbf{k}) e^{ik_\mu x^\mu} \right) \\ \phi_{,\mu}^\dagger \phi^{,\mu} &= \sum_{\mathbf{k}} \sum_{\mathbf{k}''} \frac{-1}{2V} \frac{k_\mu k''^\mu}{\sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}''}}} \left(b(\mathbf{k}) a(\mathbf{k}'') e^{-ik_\mu x^\mu} e^{-ik''_\mu x^\mu} - b(\mathbf{k}) b^\dagger(\mathbf{k}'') e^{-ik_\mu x^\mu} e^{ik''_\mu x^\mu} \right. \\ & \quad \left. - a^\dagger(\mathbf{k}) a(\mathbf{k}'') e^{ik_\mu x^\mu} e^{-ik''_\mu x^\mu} + a^\dagger(\mathbf{k}) b^\dagger(\mathbf{k}'') e^{ik_\mu x^\mu} e^{ik''_\mu x^\mu} \right) \end{aligned}$$

We have to integrate each term in \mathcal{L} over all volume to find L . When we do this to the first term $\phi_{,\mu}^\dagger \phi^{,\mu}$ above, the first sub-term on the RHS inside the parentheses immediately above will only survive if $k_i = -k''_i$. The same is true of the last sub-term. The 2nd and 3rd sub-terms will only survive if $k_i = k''_i$. So, therefore,

$$\underbrace{\int \phi_{,\mu}^\dagger \phi^{,\mu} dV}_{\text{original term in } L} = \sum_{\mathbf{k}} \frac{-k_\mu k^\mu}{2V\omega_{\mathbf{k}}} \left(b(\mathbf{k}) a(-\mathbf{k}) - b(\mathbf{k}) b^\dagger(\mathbf{k}) - a^\dagger(\mathbf{k}) a(\mathbf{k}) + a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) \right) \quad (\text{A})$$

Now, let's see what we get when we transform the spatial coordinates via $x^i \rightarrow x'^i = x^i + \alpha^i$.

$$\begin{aligned} \phi_{,\mu}^\dagger \phi^{,\mu} \xrightarrow{x^i \rightarrow x'^i = x^i + \alpha^i} &= \phi_{,\mu}^{\prime\dagger} \phi^{\prime,\mu} = \sum_{\mathbf{k}} \sum_{\mathbf{k}''} \frac{-1}{2V} \frac{k_\mu k''^\mu}{\sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}''}}} \left(b(\mathbf{k}) a(\mathbf{k}'') e^{-ik_\mu x^\mu} e^{ik_i \alpha^i} e^{-ik''_\mu x^\mu} e^{ik''_i \alpha^i} \right. \\ & \quad \left. - b(\mathbf{k}) b^\dagger(\mathbf{k}'') e^{-ik_\mu x^\mu} e^{ik_i \alpha^i} e^{ik''_\mu x^\mu} e^{-ik''_i \alpha^i} - a^\dagger(\mathbf{k}) a(\mathbf{k}'') e^{ik_\mu x^\mu} e^{-ik_i \alpha^i} e^{-ik''_\mu x^\mu} e^{ik''_i \alpha^i} \right. \\ & \quad \left. + a^\dagger(\mathbf{k}) b^\dagger(\mathbf{k}'') e^{ik_\mu x^\mu} e^{-ik_i \alpha^i} e^{ik''_\mu x^\mu} e^{-ik''_i \alpha^i} \right) \end{aligned}$$

Once again, the first and last sub-terms above, when integrated over all space, can only be non-zero if $k_i = -k''_i$, and in those cases $e^{ik_i \alpha^i} e^{ik''_i \alpha^i} = 1$. The 2nd and 3rd sub-terms will only survive if $k_i = k''_i$. In that case, $e^{ik_i \alpha^i} e^{-ik''_i \alpha^i} = 1$. When we do this, we get

$$\underbrace{\int \phi'^{\dagger}{}_{,\mu} \phi'^{\mu} dV}_{\text{transformed term in } L} = \sum_{\mathbf{k}} \frac{-k_{\mu} k^{\mu}}{2V \omega_{\mathbf{k}}} \left(b(\mathbf{k}) a(-\mathbf{k}) - b(\mathbf{k}) b^{\dagger}(\mathbf{k}) - a^{\dagger}(\mathbf{k}) a(\mathbf{k}) + a^{\dagger}(\mathbf{k}) b^{\dagger}(-\mathbf{k}) \right). \quad (\text{B})$$

Since (A) and (B) are the same, the first term in L is symmetric under the transformation.

The 2nd term in \mathcal{L}_0^0 , $-\mu^2 \phi^{\dagger} \phi$

The second term in L follows in almost identical fashion (and is simpler, since no derivatives exist in it) to the first.

$$\begin{aligned} -\mu^2 \phi^{\dagger} \phi &= -\sum_{\mathbf{k}} \sum_{\mathbf{k}''} \frac{\mu^2}{2V \sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}''}}} \left(b(\mathbf{k}) a(\mathbf{k}'') e^{-ik_{\mu} x^{\mu}} e^{-ik''_{\mu} x^{\mu}} + b(\mathbf{k}) b^{\dagger}(\mathbf{k}'') e^{-ik_{\mu} x^{\mu}} e^{ik''_{\mu} x^{\mu}} \right. \\ &\quad \left. + a^{\dagger}(\mathbf{k}) a(\mathbf{k}'') e^{ik_{\mu} x^{\mu}} e^{-ik''_{\mu} x^{\mu}} + a^{\dagger}(\mathbf{k}) b^{\dagger}(\mathbf{k}'') e^{ik_{\mu} x^{\mu}} e^{ik''_{\mu} x^{\mu}} \right) \\ \underbrace{-\int \mu^2 \phi^{\dagger} \phi dV}_{\text{original term}} &= -\sum_{\mathbf{k}} \frac{\mu^2}{2V \omega_{\mathbf{k}}} \left(b(\mathbf{k}) a(-\mathbf{k}) + b(\mathbf{k}) b^{\dagger}(\mathbf{k}) + a^{\dagger}(\mathbf{k}) a(\mathbf{k}) + a^{\dagger}(\mathbf{k}) b^{\dagger}(-\mathbf{k}) \right). \quad (\text{C}) \end{aligned}$$

When we transform the spatial coordinates via $x^i \rightarrow x'^i = x^i + \alpha^i$, we get

$$\begin{aligned} -\mu^2 \phi^{\dagger} \phi \xrightarrow{x^i \rightarrow x^i = x'^i - \alpha^i} &= -\mu^2 \phi'^{\dagger} \phi' = -\sum_{\mathbf{k}} \sum_{\mathbf{k}''} \frac{\mu^2}{2V \sqrt{\omega_{\mathbf{k}} \omega_{\mathbf{k}''}}} \left(b(\mathbf{k}) a(\mathbf{k}'') e^{-ik_{\mu} x'^{\mu}} e^{ik_i \alpha^i} e^{-ik''_{\mu} x'^{\mu}} e^{ik''_i \alpha^i} \right. \\ &\quad \left. + b(\mathbf{k}) b^{\dagger}(\mathbf{k}'') e^{-ik_{\mu} x'^{\mu}} e^{ik_i \alpha^i} e^{ik''_{\mu} x'^{\mu}} e^{-ik''_i \alpha^i} + a^{\dagger}(\mathbf{k}) a(\mathbf{k}'') e^{ik_{\mu} x'^{\mu}} e^{-ik_i \alpha^i} e^{-ik''_{\mu} x'^{\mu}} e^{ik''_i \alpha^i} \right. \\ &\quad \left. + a^{\dagger}(\mathbf{k}) b^{\dagger}(\mathbf{k}'') e^{ik_{\mu} x'^{\mu}} e^{-ik_i \alpha^i} e^{ik''_{\mu} x'^{\mu}} e^{-ik''_i \alpha^i} \right). \end{aligned}$$

When we integrate the above over space, the same sub-terms will drop out in the same way as did to get (B). Thus, we end up with

$$\underbrace{-\int \mu^2 \phi^{\dagger} \phi dV}_{\text{transformed term}} = -\sum_{\mathbf{k}} \frac{\mu^2}{2V \omega_{\mathbf{k}}} \left(b(\mathbf{k}) a(-\mathbf{k}) + b(\mathbf{k}) b^{\dagger}(\mathbf{k}) + a^{\dagger}(\mathbf{k}) a(\mathbf{k}) + a^{\dagger}(\mathbf{k}) b^{\dagger}(-\mathbf{k}) \right). \quad (\text{D})$$

Since (C) and (D) are the same, the second term in L is also symmetric under the transformation, and thus L is symmetric under it.

From macro variational mechanics, we know that if L is symmetric in some coordinate, then the conjugate momentum of that coordinate is conserved. k_i , the particle(s) 3-momentum is the conjugate momentum of x^i . Thus, k_i is conserved. Note one subtlety. To get to macro mechanics we integrated over all field coordinates x^i , so there was no x^i coordinate left in L . Macroscopically, we would then need to consider our x^i coordinate as that of the position of the center of mass of our solid body (particle). A transformation on the field coordinates x^i would then be the same transformation on the center of mass x^i coordinate used in macro, solid body, variational mechanics analysis.

Ans. (second part):

$$H = \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(N_a(\mathbf{k}) + N_b(\mathbf{k}) \right) \quad \mathbf{P} = \sum_{\mathbf{k}} \mathbf{k} \left(N_a(\mathbf{k}) + N_b(\mathbf{k}) \right) \quad \rightarrow \quad [H, \mathbf{P}] = 0 \quad \left(\begin{array}{l} \text{because all number} \\ \text{operators commute} \end{array} \right)$$

Thus \mathbf{P} is conserved for the free Hamiltonian.