Chapter 6 Problem Solutions

Prob. 15, Corrected version of 2nd edition.

Use the transformation $x^0 \to x'^0 = x^0 + \alpha$ for free scalars to show that energy $\omega_{\mathbf{k}}$ is conserved. Hint: There are subtle issues (both in classical and quantum field theory) with using the Lagrangian to show this. Instead, show it directly from the Hamiltonian density. Is it immediately obvious that you will get the same results from commutation of the energy operator with the Hamiltonian? (Tricky wording here?)

Ans. (first part).

Consider the Hamiltonian directly, instead of the Lagrangian, and refer to Sect. 3.4.1, pgs. 53-54.

$$\mathcal{H}_0^{\;0} = \dot{\phi}^\dagger \dot{\phi} + \phi^\dagger{}_{,i}\phi_{,i} + \mu^2 \phi^\dagger \phi \qquad \text{(G)}$$

$$\underbrace{\int \left(\dot{\phi}^\dagger \dot{\phi} + \phi^\dagger{}_{,i}\phi_{,i} + \mu^2 \phi^\dagger \phi\right) dV}_{\text{original terms in } H} = \sum_{\mathbf{k}} \frac{\omega_{\mathbf{k}}^2}{2\omega_{\mathbf{k}}} \left(-a(-\mathbf{k}) b(\mathbf{k}) e^{-i2\omega_{\mathbf{k}} t} + a(\mathbf{k}) a^\dagger(\mathbf{k}) + b^\dagger(\mathbf{k}) b(\mathbf{k}) - b^\dagger(-\mathbf{k}) a^\dagger(\mathbf{k}) e^{i2\omega_{\mathbf{k}} t} \right)$$
$$+ \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{2\omega_{\mathbf{k}}} \left(b(\mathbf{k}) a(-\mathbf{k}) e^{-2i\omega_{\mathbf{k}} t} + a^\dagger(\mathbf{k}) a(\mathbf{k}) + b(\mathbf{k}) b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) e^{2i\omega_{\mathbf{k}} t} \right) \qquad \text{(H)}$$
$$+ \sum_{\mathbf{k}} \frac{\mu^2}{2\omega_{\mathbf{k}}} \left(b(\mathbf{k}) a(-\mathbf{k}) e^{-2i\omega_{\mathbf{k}} t} + b(\mathbf{k}) b^\dagger(\mathbf{k}) + a^\dagger(\mathbf{k}) a(\mathbf{k}) + a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) e^{2i\omega_{\mathbf{k}} t} \right).$$

$$\text{original } H = \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} \begin{pmatrix} \left(-\omega_{\mathbf{k}}^2 a(-\mathbf{k}) b(\mathbf{k}) e^{-i2\omega_{\mathbf{k}} t} + \omega_{\mathbf{k}}^2 a(\mathbf{k}) a^\dagger(\mathbf{k}) + \omega_{\mathbf{k}}^2 b^\dagger(\mathbf{k}) b(\mathbf{k}) - \omega_{\mathbf{k}}^2 a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) e^{i2\omega_{\mathbf{k}} t} \right) \\ + \sum_{\mathbf{k}} \left(\mathbf{k}^2 a(-\mathbf{k}) b(\mathbf{k}) e^{-2i\omega_{\mathbf{k}} t} + \mathbf{k}^2 a^\dagger(\mathbf{k}) a(\mathbf{k}) + \mathbf{k}^2 b(\mathbf{k}) b^\dagger(\mathbf{k}) + \mathbf{k}^2 a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) e^{2i\omega_{\mathbf{k}} t} \right) \\ + \sum_{\mathbf{k}} \left(\mu^2 a(-\mathbf{k}) b(\mathbf{k}) e^{-2i\omega_{\mathbf{k}} t} + \mu^2 b(\mathbf{k}) b^\dagger(\mathbf{k}) + \mu^2 a^\dagger(\mathbf{k}) a(\mathbf{k}) + \mu^2 a^\dagger(\mathbf{k}) b^\dagger(-\mathbf{k}) e^{2i\omega_{\mathbf{k}} t} \right) \end{pmatrix}$$
$$= \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} \begin{pmatrix} \underbrace{\left(-\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + \mu^2\right)}_{=0} a(-\mathbf{k}) b(\mathbf{k}) e^{-i2\omega_{\mathbf{k}} t} + \underbrace{\left(\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + \mu^2\right)}_{2\omega_{\mathbf{k}}^2} a^\dagger(\mathbf{k}) a(\mathbf{k}) + \omega_{\mathbf{k}}^2 \\ + \underbrace{\left(\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + \mu^2\right)}_{2\omega_{\mathbf{k}}^2} b^\dagger(\mathbf{k}) b(\mathbf{k}) + \underbrace{\left(\mathbf{k}^2 + \mu^2\right)}_{\omega_{\mathbf{k}}^2} + \underbrace{\left(-\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + \mu^2\right)}_{=0} a(-\mathbf{k}) b(\mathbf{k}) e^{i2\omega_{\mathbf{k}} t} \end{pmatrix} \qquad \text{(I)}$$
$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + \tfrac{1}{2} + b^\dagger(\mathbf{k}) b(\mathbf{k}) + \tfrac{1}{2} \right).$$

After the transformation $x^0 \to x'^0 = x^0 + \alpha$, we will get an extra factor of $e^{\pm i2\omega_{\mathbf{k}}(-\alpha)}$ in the terms dependent on t in the middle of (I).

$$\text{transformed } H = \sum_{\mathbf{k}} \frac{1}{2\omega_{\mathbf{k}}} \begin{pmatrix} \underbrace{\left(-\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + \mu^2\right)}_{=0} a(-\mathbf{k}) b(\mathbf{k}) e^{-i2\omega_{\mathbf{k}}(t-\alpha)} + \underbrace{\left(\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + \mu^2\right)}_{2\omega_{\mathbf{k}}^2} a^\dagger(\mathbf{k}) a(\mathbf{k}) + \omega_{\mathbf{k}}^2 \\ + \underbrace{\left(\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + \mu^2\right)}_{2\omega_{\mathbf{k}}^2} b^\dagger(\mathbf{k}) b(\mathbf{k}) + \underbrace{\left(\mathbf{k}^2 + \mu^2\right)}_{\omega_{\mathbf{k}}^2} + \underbrace{\left(-\omega_{\mathbf{k}}^2 + \mathbf{k}^2 + \mu^2\right)}_{=0} a(-\mathbf{k}) b(\mathbf{k}) e^{i2\omega_{\mathbf{k}}(t-\alpha)} \end{pmatrix}$$
$$= \sum_{\mathbf{k}} \omega_{\mathbf{k}} \left(a^\dagger(\mathbf{k}) a(\mathbf{k}) + \tfrac{1}{2} + b^\dagger(\mathbf{k}) b(\mathbf{k}) + \tfrac{1}{2} \right).$$

But those terms equal zero due to their numeric coefficients. Therefore, the Hamiltonian is conserved under a time translation.

Ans. (second part). The question is a trick, since the energy operator is the Hamiltonian, and any operator must commute with itself. Thus, energy is conserved via the "commutation with the Hamiltonian means conserved" rule.