Chapter 6 *Version date: December 27, 2011*

Symmetry, Invariance, and Conservation for Free Fields

"The time has come", the walrus said, "to speak of many things, of symmetries, Lagrangians, and changeless transformings."

Re-rendering of Lewis Carroll
by R. Klauber

6.0 Preliminaries

My apologies to Lewis Carroll for the liberties taken with his great work, but the Jabberwockian, oxymoron-like phrase "changeless transforming" will come to have deep significance for us. We will find it central to our understanding of symmetry in general, and more specifically, in our study of quantum field theory.

6.0.1 Background

Symmetry is one of the most aesthetically captivating and philosophically meaningful concepts known to mankind. Rooted originally in the arts, it has evolved and re-emerged in our modern age as a unified and holistic structural basis for all of science.

But if so, what then, particularly in mathematical terms, is it? If, in a work of art, it is a quality, perhaps somewhat abstract and related closely to feeling and emotion, how does it relate to physics? Can it be defined precisely?

We begin our answer to these questions after the chapter preview below.

6.0.2 Chapter Overview

A simple definition of symmetry with examples

First, an introduction to symmetry,
where we will look at

- a simple definition of symmetry without math,
- examples of symmetry, and
- a mathematical definition of symmetry.

Symmetry in classical mechanics

Then, symmetry in classical physics, including

- laws of nature symmetric under Lorentz transformation, i.e., laws are invariant in spacetime (same for all observers)
- symmetry in the Lagrangian $L \rightarrow$ a related quantity is conserved

Symmetry in QFT

Then, symmetry in quantum field theory, including

- field equations symmetric under Lorentz transformation, i.e., they are invariant in spacetime (same for all observers)
- symmetry in the Lagrangian density $\mathcal{L} \rightarrow$ a related quantity is conserved

- symmetry, gauges, and gauge theories

Free vs interacting fields

We will deal primarily with free particles and fields in this chapter, but the principles will apply in general, as we shall see when we investigate interactions.

Symmetry principles apply to free and interacting cases, but only free in this chapter

6.1 Introduction to Symmetry

6.1.1 Symmetry Simplified

Each of us has some intuitive feel for what symmetry is, though most might, at least at first, have some difficulty coming up with a very precise definition. Certainly snowflakes have symmetry, and so do cylinders and beach balls. A map of New York probably does not. Just what exactly is it that we sense about an object that causes us to deem it symmetric?

To see what that certain something is, imagine yourself looking at a real life version of the cylinder depicted in the figure below. Then imagine closing your eyes for a moment, and during the time you can't see, someone else rotates the cylinder about the vertical axis shown in the figure. When you open your eyes is there any way you could tell that the rotation had taken place? The answer, of course, is no, but what does that mean?

It means that even though something changed (the rotational position of the cylinder), something else remained unchanged. The form we perceive, the wholeness that is the cylinder, looks exactly the same. The act of moving or "transforming" the cylinder simultaneously exhibits the qualities of both change (transformation) and non-change (invariance).

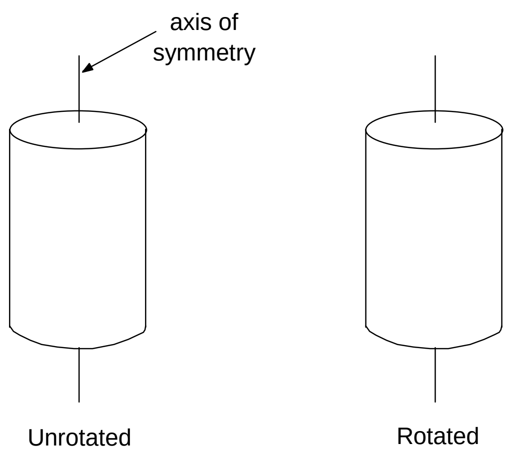

Figure 6-1. Symmetry of a Cylinder

So what then is *symmetry*? It is simply the propensity *for non-change with change, for invariance under transformation.* In many cases, such as this one, it is a relationship between the whole and the parts in which the whole can exhibit changelessness while the component parts change. In virtually every case, it involves superficial change with more profound non-change.

Symmetry is the propensity for non-change with change

Symmetry manifests to greater or lesser degrees. A sphere, for instance, has more symmetry than a cylinder because it possesses innumerable (rather than only one) possible axes about which it could be rotated and still appear the same. A snowflake has even less symmetry than a cylinder, since there are only six discrete positions into which it could be rotated where no change could be discerned. A baseball glove has no symmetry whatever. There are absolutely no ways it could be rotated without looking distinctly different.

Different degrees of non-change with change mean different degrees of symmetry

Symmetry extends beyond rotation. Consider an infinite length horizontal line. Translate it 10 meters to the right. It still looks the same. It has translational symmetry. Consider the human body where the right half is reflected to the left, and the left half reflected to the right. It still looks the same (to good approximation.) To high degree, our bodies have mirror, or reflection, symmetry.

Different kinds of symmetry: rotational, translational, refection

There are continuous symmetries, like the cylinder of Fig. 6-1, a sphere, or the infinite straight line discussed above. For these, transformation is continuous. And there are discrete symmetries, like the snowflake, an infinite picket fence, or any reflection symmetry. For these, the transformation only maintains an invariant quality in certain discrete positions.

Symmetries can be continuous or discrete

Extrapolating these ideas beyond mere geometry and rotation, we can begin to understand why symmetry is considered so meaningful and fascinating. Non-change with change permeates many diverse phenomena. In many works of visual art, such as those of Escher or Indian mandalas, this principle is evident. In architecture, it has been pervasive throughout the ages. In music, the refrain, typically the essence of a song, remains the same, while other lyrics change. And that certain something we sense in the work of a great master is typically there throughout all of his or her individual pieces. We know that a Bach sonata, even if we have never heard it before, is by Bach. We know a Picasso painting, even if we have never seen it before, is by Picasso. We sense symmetry.

Symmetry plays a major role in the arts and elsewhere

6.1.2 Symmetry Mathematically

In mathematical terms, the rotations, translations, and reflections we discussed in the previous section are known as transformations. Any transformation, by definition, is a change of something. If the transformation is symmetric, something else remains unchanged, or in math terms, invariant. Not all transformations are symmetric, of course. We will look at some mathematical examples below, but first we need to note one more thing.

Mathematically, symmetry comprises invariance under transformation

The transformation depicted in Fig. 6-1 can be understood either as a rotation of the cylinder in one direction while we remain fixed (an active transformation, by name), or alternatively, as a rotation of our viewing frame of reference in the other direction while the cylinder remains fixed (a passive transformation). The same thing is true for snowflakes, the translation of a straight line, and more. Transformations typically involve a *change of perspective*, a change in the relationship between the observer and the thing being observed.

Transformation is change of object with observer fixed or vice versa.

Mathematically, when we change our position of observation, it is equivalent to using a new, different reference frame and coordinate system, oriented differently from, and/or displaced relative to, the original. So a transformation can be viewed simply as a change of coordinate system, and this is often represented as a shifting from unprimed to primed coordinates. We will focus on this (passive transformation) interpretation, the most common one in physics, and most relevant to QFT.

Changing observer = changing coordinate system, most useful interpretation for us

Example #1

So how about some simple examples? For starters, see Fig. 6-2, where on the left hand side we show the function

$$f\left(x^1,x^2\right)=\left(x^1\right)^2+\left(x^2\right)^2 . \qquad (6\text{-}1)$$

Example of a function symmetric under rotation transformation

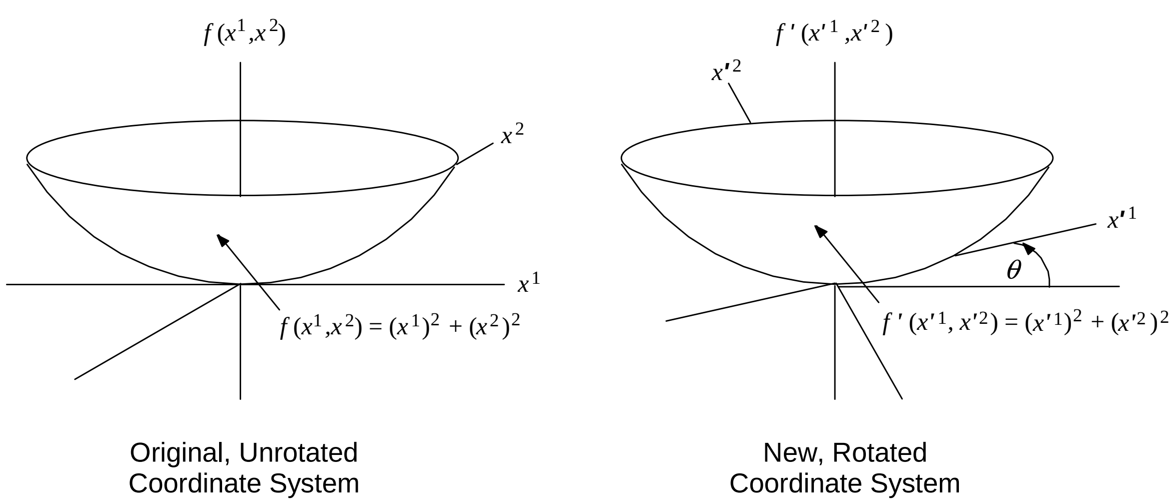

Figure 6-2. Example of a Function Symmetric Under Coordinate Transformation

We then change to a coordinate system rotated relative to the first, where our transformation from the first set of coordinates to the second is

$$x'^1 = x^1 \cos\theta + x^2 \sin\theta \qquad x'^2 = -x^1 \sin\theta + x^2 \cos\theta\,, \tag{6-2}$$

with the inverse transformation being

$$x^1 = x'^1 \cos\theta - x'^2 \sin\theta \qquad x^2 = x'^1 \sin\theta + x'^2 \cos\theta\,. \tag{6-3}$$

In matrix form, these are

$$\begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}}_{T} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \qquad \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{T^{-1}=T^{\mathrm{T}}} \begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix}, \tag{6-4}$$

2D rotation transformation

where we designate the transformation by T, whose inverse is its own transpose.

Substituting (6-3) into (6-1) to express our function in the new system primed coordinates yields

$$f\left(x^1,x^2\right) = \left(x^1\right)^2 + \left(x^2\right)^2 =$$

$$f'\left(x'^1,x'^2\right) = \left(x'^1 \cos\theta - x'^2 \sin\theta\right)^2 + \left(x'^1 \sin\theta + x'^2 \cos\theta\right)^2 \tag{6-5}$$

$$= \left(x'^1\right)^2\left(\cos^2\theta + \sin^2\theta\right) + \left(x'^2\right)^2\left(\cos^2\theta + \sin^2\theta\right) = \left(x'^1\right)^2 + \left(x'^2\right)^2 = f\left(x'^1,x'^2\right).$$

Function has same form in original or primed coordinates → it is symmetric under the transformation

The function has exactly the same form in both coordinate systems, exactly the same form whether we express it in terms of the unprimed or primed coordinates. Given Fig. 6-2, this should not be much of a surprise.

The prime on f' is used to indicate it has, in general, different functional form from f, which is the case for non-symmetric functions. But since the function f here is symmetric under the transformation, the functional form of f and f' is the same, so we drop the prime. This can be more easily understood with the following example.

Example #2

Consider the function

$$g\left(x^1,x^2\right) = \left(x^2\right)^2. \tag{6-6}$$

Example of function not symmetric under rotation transformation

Express (6-6) in the primed coordinate system by substituting (6-3) into it, and we get

$$g = \left(x^2\right)^2 = \left(x'^1 \sin\theta + x'^2 \cos\theta\right)^2 = \left(x'^1\right)^2 \sin^2\theta + \left(x'^2\right)^2 \cos^2\theta + 2x'^1 x'^2 \sin\theta\cos\theta \neq \left(x'^2\right)^2. \tag{6-7}$$

Thus, g has different form in the two systems and is *not* symmetric under the transformation T.

$$g\left(x^1,x^2\right) = g'\left(x'^1,x'^2\right) \neq g\left(x'^1,x'^2\right) \text{ but } f\left(x^1,x^2\right) = f'\left(x'^1,x'^2\right) = f\left(x'^1,x'^2\right). \tag{6-8}$$

The transformed form of g, represented by g', has the same value at the same physical point, but it is not the same form in terms of the primed coordinates as g was in terms of the unprimed coordinates. But f', the transformed form of f, did have the same form in terms of both sets of coordinates, and thus, we dropped the prime on f on the RHS of (6-8).

In spite of its non-symmetry under rotation, g is symmetric under a different kind of transformation, the translation to a coordinate system which is displaced relative to the first along the x^1 axis, i.e., $x^1 \rightarrow x'^1 = x^1$ + constant, or

$$\begin{bmatrix} x'^1 \\ x'^2 \end{bmatrix} = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} K \\ 0 \end{bmatrix} \qquad K = \text{constant}\,. \tag{6-9}$$

But same function is symmetric under a translation transformation

Substitution of (6-9) into (6-6) yields $g'(x'^1, x'^2) = (x'^2)^2$, having the same form in both systems.

Lessons from the Examples

If a coordinate is missing from f, then f is symmetric with respect to change of that coord

From Example #2, we can deduce the general rule that if a coordinate is missing in a given function, that function is invariant under a transformation solely in the direction of that coordinate (and also under multiplication of the coordinate by a constant, which will be less important for us.) The function is symmetric with respect to that transformation.

In both examples and in general, the value of a particular function at a given physical point in space is the same under any transformation, symmetric or not. The new coordinates are simply a new way to designate that particular point with different numbers, but it is the same point in space, and hence must have the same numeric value for functional there. If f or g were a physical entity, like pressure, simply changing our coordinates would not change the value of the pressure at any given point in space, even though the numbers describing that point's location are different.

Value of a scalar function at a physical point is the same in any coordinate system

So under *any* transformation of coordinate axes, the value at a physical point of every possible scalar function is invariant. Under a *symmetry* transformation the form of the function also is invariant. Under a non-symmetry transformation, the form of the function looks different in terms of the new coordinates, and we represent that functional difference with a prime on the function label.

Functional form of a scalar is the same under a symmetry transformation

Scalars are Invariant, Vectors are Covariant

Consider a 2D position vector in physical space represented in the unprimed coordinates of Example #1 by $x^i = (x^1, x^2)$. Under the rotation transformation T, this becomes $x'^i = (x'^1, x'^2) \neq (x^1, x^2)$. A different (i.e., non-invariant) set of coordinates represents the exact same vector. But it is the same vector at the same physical location, and in fact, has the same length in each coordinate system equal to

Vector components change under transformation

$$|\mathbf{x}| = |x^i| = \sqrt{(x^1)^2 + (x^2)^2} = \sqrt{(x'^1)^2 + (x'^2)^2} = |x'^i|. \qquad (6\text{-}10)$$

But vector length and direction in physical space unchanged for any coordinate system

So the scalar value at the point (equal to the length of the position vector at that point) is the same in both systems, but the coordinate values are not.

It is generally true of every vector **v**, not just the position vector shown here, that its physical, measurable length (a scalar value) remains unchanged under any coordinate transformation, but its component values change. This is called covariance. Scalar values are invariant under coordinate transformation; vector components are covariant. (Don't confuse this use of the word "covariant" with our use of the terms covariant and contravariant coordinates.)

Vectors are covariant under coordinate transformation

Parallel to scalars, if the vector components remain unchanged under a given transformation, then that transformation is a symmetry transformation, i.e., $v'^j(x'^i) = v^j(x^i)$. One example is the **E** field around a point charge, which points radially outward from the point, described in a coordinate system with origin at the point. Rotating to a new coordinate system, we find the same functional dependence of the **E** field on the new coordinates. See Prob. 7.

Vector transformation symmetric if components unchanged

All of these conclusions are valid for any dimension space, and in particular for our purposes, the 4D spacetime of relativity theory. They are also valid for systems of generalized coordinates (reviewed in Chap. 2), not just Cartesian like those shown here, and for both particles and fields. Probs. 1 through 6 and Wholeness Chart 6-1 can help you gain more comfort with these concepts.

All of above true for 4D *and other spaces, as well*

Wholeness Chart 6-1. Symmetry Summary

	Non-Symmetric Transformation	**Symmetric Transformation**
Coordinate values change?	Yes	Yes
Scalar value at a physical point the same?	Yes	Yes
Form of function invariant?	No	Yes
Vector magnitude and direction at a physical point the same?	Yes	Yes
Vector components invariant?	No	Yes
Vector components vary covariantly?	Yes	No, invariant
General rule: If a function h is not a function of the jth coordinate x^j, then h is symmetric under the transformation $x^j \rightarrow x^j$ + constant		

6.2 Symmetry in Classical Mechanics

6.2.1 Invariance of the Laws of Nature

Einstein showed laws of nature symmetric (invariant) in spacetime

Symmetry turns out to play an extraordinary role in the physics of our creation. Albert Einstein, in possibly the most far reaching of any scientific discovery, provided the first insight into the universe's innate symmetry. He showed, via his theories of relativity, that even though the visible world of changing objects appears different at different places, in different times, to different observers, the physical laws of nature governing those objects remain invariant regardless of when, where, or how they are perceived. The laws of physics, acting on a subtler, more holistic level of creation, exhibit changelessness in the midst of change and are said to be *symmetric* throughout spacetime.

We review this discovery by Einstein and the classical mechanics leading up to it below. This should not be new material for most readers, but since it forms a good part of the foundation for understanding the ramifications of symmetry in physics, I provide the following overview, which many readers are probably well versed enough in to skim, or skip, over.

6.2.2 Brief History of Einstein's Insight into Symmetry

<u>Newton's Laws: Invariant under Galilean Transformations</u>

F=*m***a** *invariant under rotation and translation*

Newton's laws of motion are trivially symmetric under a rotation transformation because **F**=*m***a** (from which the other two laws can be derived) is a vector equation, and as we showed above, any vector maintains its magnitude and direction unchanged in physical space under rotation. So the **F** and **a** vectors will still be aligned with each other in any new coordinate system and still have the same proportionality constant of *m*. There will be new coordinates for each, but the same equation relating those coordinates will hold. If $F^i = ma^i$, then $F'^i = ma'^i$. Similar results hold for a translation transformation, which you can show by doing Prob. 8.

The (3D) Galilean transformation is to coordinate system having constant velocity relative to original system

Another type of transformation involves changing from one coordinate system to another where the transformed system has a constant velocity relative to the first of **v**. Any fixed coordinate value in the original system appears to move in the – **v** direction relative to the second. This, as most readers should know, is called a <u>Galilean transformation</u> for the 3D plus time of classical mechanics and is

$$\begin{bmatrix} x^1 \\ x^2 \\ x^3 \end{bmatrix} \rightarrow \begin{bmatrix} x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} x^1 - v^1 t \\ x^2 - v^2 t \\ x^3 - v^3 t \end{bmatrix} \qquad \text{or} \qquad \mathbf{x} \rightarrow \mathbf{x} - \mathbf{v}t\,. \tag{6-11}$$

In Newtonian/Galilean mechanics, time does not change from one system to the other. It is invariant and thus labeled by *t* in both systems above.

F=*m***a** *invariant under Galilean transformation*

Newton's second law is invariant under this transformation because of the second order time derivative in x^i. That is,

$$\begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} = m\begin{bmatrix} \ddot{x}^1 \\ \ddot{x}^2 \\ \ddot{x}^3 \end{bmatrix} \rightarrow \; = m\begin{bmatrix} \ddot{x}'^1 \\ \ddot{x}'^2 \\ \ddot{x}'^3 \end{bmatrix} = m\frac{d^2}{dt^2}\begin{bmatrix} x^1 - v^1 t \\ x^2 - v^2 t \\ x^3 - v^3 t \end{bmatrix} = m\begin{bmatrix} \ddot{x}^1 \\ \ddot{x}^2 \\ \ddot{x}^3 \end{bmatrix} = \begin{bmatrix} F^1 \\ F^2 \\ F^3 \end{bmatrix} \quad \text{or} \quad \mathbf{F} = m\mathbf{a} \rightarrow \mathbf{F} = m\mathbf{a}. \tag{6-12}$$

Prior to Maxwell's appearance on the scene, it was generally assumed that all laws of nature were invariant under Galilean transformations.

<u>Maxwell's Laws: Invariant under a Different Kind of Transformation</u>

Maxwells' equations NOT invariant under Galilean transformation

However, with the publication of Maxwell's equations in 1864, it was realized that his laws of nature, in contrast, do not transform symmetrically under a Galilean transformation. If one invoked (6-11) in the coordinates of Maxwell's equations, the result was a set of equations of different form, quite unlike the behavior that we saw in (6-12). That exercise is fairly involved and would lead us too far afield, so we won't get into it here.

It was, however, realized that Maxwell's equations were invariant under a different transformation between coordinate systems in relative motion. This transformation is 4D with time and space transformations, rather than simply 3D spatial, and is called the Lorentz transformation, after its discoverer. It is, where we lose no generality by restricting relative velocity to a single coordinate direction (since Maxwell's laws are symmetric under rotation), and where we write out c in non-natural units this one time,

Maxwells' equations ARE invariant under Lorentz transformation

$$\begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \to \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \begin{bmatrix} \gamma\left(x^0 - \frac{v}{c}x^1\right) \\ \gamma\left(x^1 - \frac{v}{c}x^0\right) \\ x^2 \\ x^3 \end{bmatrix} = \underbrace{\begin{bmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\Lambda} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \qquad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad v = v^1 . \quad (6\text{-}13)$$

The (4D) Lorentz transformation

Again, because it would lead us away from our present tasks, we will not go through the lengthy process of showing that Maxwell's equations retain the same form under (6-13). Note that we will designate the Lorentz transformation with the symbol Λ.

Einstein's Resolution

Einstein and others wanted all equations of nature to display invariance under the same transformation. There were two major sets of laws at the time, Newton's for mechanics and Maxwell's for electromagnetism. But they didn't transform to moving coordinates in the same way. Something had to give.

Scientists wanted all laws symmetric under same transformation

Einstein intuited that the speed of light must be the same for all observers, whether they are fixed relative to one another or have relative constant velocity. This was quite a radical insight and turned out to be true. Since the wave solution to Maxwell's equations yielded a speed of e/m waves (light) of c, that meant those equations must yield the same result in any coordinate system that was not accelerating (nor in a gravitational field). To do this, they must have the same form in all such coordinate systems. The only transformation that did that was Lorentz's.

Einstein figured c = same for all observers

This meant Maxwell's equations same for all

Einstein took that "to the bank". He knew it meant that in order for the equations of mechanics to also be symmetric under the Lorentz transformation, they must be modified. Newton's laws were not the exact truth, but only a very good approximation under normal human conditions to more accurate and precise laws.

So Lorentz transformation must be correct one for e/m and mechanics laws

We won't write Einstein's law of mechanics here, but refer interested readers to any textbook on relativity[1]. The point is that his reformulation of mechanics 1) is invariant in form under Lorentz transformations, and 2) reduces to high accuracy, at speeds for objects of much less than c, to Newton's 2nd law.

Einstein reformulated mechanics so it obeyed Lorentz transformation

We summarize these results in Wholeness Chart 6-2.

That reformulation is special relativity

Wholeness Chart 6-2. Galilean vs Lorentz Transformations

	Galilean transformation	**Lorentz transformation**
Newton's laws symmetric?	Yes	No
Maxwell's laws symmetric?	No	Yes
Einstein law of mechanics symmetric?	No	Yes
Valid for any speed?	No	Yes
Valid at low speed?	Yes, approximately	Yes

Special relativistic mechanics becomes classical mechanics at v << c (<< 1 *in nat units*)

[1] For example, Hartle, James B., *Gravity*, Pearson (2003), Chap. 5

Einstein carried this idea further in the development of his general theory of relativity. In very general terms, the same concept holds. At any given point in spacetime, the laws of nature, expressed as relationships between physical entities (like scalars, vectors, and tensors) are invariant in form. However, the Lorentz transformation is specifically for differences in velocity in non-accelerating, non-gravitational, systems. All of our work in this text will assume acceleration and gravitational effects are zero or small enough to be ignored.

Our work special relativity, not general relativity

The bottom line: All laws of nature are symmetric (invariant) under Lorentz transformation[2]. They are the same for all observers in relative constant velocity motion.

Laws of nature symmetric under Lorentz transformation

6.2.3 More with Lorentz Transformations

Index notation for Lorentz transformations

The relation (6-13) expressed in shorthand notation for the position vector, and the corresponding transformations for a four vector V^μ and a 4D tensor $T^{\mu\nu}$ (for those familiar with tensors) can be written as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \qquad V'^\mu\left(x'^\alpha\right) = \Lambda^\mu{}_\nu V^\nu\left(\Lambda^\alpha{}_\beta x^\beta\right) \qquad T'^{\mu\nu}\left(x'^\alpha\right) = \Lambda^\mu{}_\delta \Lambda^\nu{}_\gamma T^{\delta\gamma}\left(\Lambda^\alpha{}_\beta x^\beta\right). \qquad (6\text{-}14)$$

Note that Λ^{-1}, the inverse of Λ, can be obtained by taking $\mathbf{v} \rightarrow -\mathbf{v}$ in (6-13) since each coordinate system seems to be going in the opposite direction with respect to the other. Λ^{-1} will transform x'^μ back into x^μ.

Length of any four-vector invariant

Recall from (6-10) that the length of a vector in 3D is unchanged under a coordinate system transformation, i.e., the length is a scalar and thus invariant. The same thing is true in 4D for four-vectors. Recall further, from Chap. 2, particularly the appendix Sect. XXX 2.9.3, pg 35 XXX, that the length of any four-vector, symbolized by w^μ, is the square root of the 4D inner product, i.e., of

Length of any 4D vector invariant under Lorentz transformation

$$w_\mu w^\mu = w_0 w^0 + w_1 w^1 + w_2 w^2 + w_3 w^3 = w^0 w^0 - w^1 w^1 - w^2 w^2 - w^3 w^3 = \text{scalar invariant} \qquad (6\text{-}15)$$

and that this is the same for any observer in any coordinate system. This applies to any vector, be it a position vector like x^μ, the differential of position dx^μ, the four-velocity u^μ of the Chap. 2 appendix, the four-potential A^μ, the partial derivative ∂^μ, or any other.

Do Prob. 9 to show that under a Lorentz transformation (6-15) is invariant and therefore is Lorentz invariant. We often call such scalars in 4D world scalars or Lorentz scalars.

4D Laplacian derivative invariant

Similarly, just as $x_\mu x^\mu$ is Lorentz invariant, so is the differential length squared $dx_\mu dx^\mu$. Putting the latter into the denominator, we get the corresponding 4D Laplacian derivative

4D Laplacian invariant under Lorentz transformation

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \partial_\mu \partial^\mu = \frac{\partial}{\partial x^0}\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}\frac{\partial}{\partial x^2} - \frac{\partial}{\partial x^3}\frac{\partial}{\partial x^3} = \text{ scalar invariant derivative}, \qquad (6\text{-}16)$$

which, as noted, is Lorentz invariant itself. So, where X represents any quantity,

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} = \frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x'_\mu} \quad \rightarrow \quad \frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu} X = \frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x'_\mu} X\ . \qquad (6\text{-}17)$$

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[2] I don't want to confuse readers, but most specialists in relativity would bring up caveats here. For one, in systems with clock synchronization done under a different convention than Einstein's, the laws of nature actually do take different form. The Lorentz transformation assumes Einstein synchronization. We will stick with that, the simplest, most efficient, synchronization and with the most widely used transformation of Lorentz. Our statements with regard to symmetries thereunder will hold true in general for our work and are widely accepted as valid criteria, which good theories should meet.

However, a second caveat involves research being done at the time of this writing that questions whether Lorentz symmetry needs to be upheld in certain very advanced theories of elementary particles.

Please do not worry about these things now. You can do so, when and if your work leads you into these other areas.

In general, any time we sum over pairs of indices, even if the factors in the summation are partial derivatives, we get a scalar invariant as a result.

Most general transformation between 4D coordinates

The most general transformation we could have in spacetime would comprise 1) a 4D translation (translating our coordinate axes in space, time, or both), 2) a rotation in space, and 3) a Lorentz transformation to a frame with different relative velocity. (We ignore reflection.)

The rotation in 3D is the same as we should have seen in classical mechanics and thus, for our purposes, is not of great interest. It does allow us to rotate our 3D axes, however, so that the relative velocity between our original and transformed systems is along the x^1 axes of both. This lets us use the Lorentz transformation in its simplest form, the Λ of (6-13). With this form we state the general transformation between coordinate systems, known as the Poincaŕe transformation as

Poincaŕe transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\left(x^{\nu} + a^{\nu}\right) \qquad a^{\nu} = \text{constant four vector} . \tag{6-18}$$

Our laws of nature invariably involve partial derivatives with respect to time and space, i.e, with respect to infinitesimal differences in 4D position. These differences between coordinates in any given coordinate system are unchanged by displacing the coordinates by any constant a^{ν}. So the laws of nature will not be changed by a translation of the 4D coordinate system by a^{ν}. The laws are the same at any place x^i, and at any time x^0.

Laws of nature symmetric under Poincare' transformation

Bottom line: Thus, you may hear it said sometimes that the laws of nature are invariant with respect to Poincare' transformations.

6.2.4 Other Kinds of Symmetry

There are other kinds of symmetry, other than that of Lorentz symmetry in spacetime. You should have studied this, at least to some degree, in classical mechanics, but we will review the essence of it here.

Other kinds of symmetry exist

Symmetry of the Lagrangian Implies a Conserved Quantity

Consider the Euler-Lagrange equation for a particle in Newtonian mechanics

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0 \qquad L = T - V \qquad p_i = \frac{\partial L}{\partial \dot{x}^i} . \tag{6-19}$$

If the Lagrangian L is not an explicit function of the spatial coordinate x^i, then $\partial L/\partial x^i = 0$ on the LHS above. Thus, the time derivative of p_i is zero.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = \frac{dp_i}{dt} = 0 \quad \text{with} \quad \frac{\partial L}{\partial x^i} = 0 \quad \text{when} \quad L \neq L\left(x^i\right) . \tag{6-20}$$

Hence p_i is constant and thus, conserved. This makes sense since the only source for spatial dependence in L is the potential energy V. The gradient of potential energy $-\partial V/\partial x^i$ is force on the particle. If we have no V dependence on x^i, then there is no force in the x^i direction, and momentum p_i is constant. For example, if V is a function of x^1, but not of x^2 or x^3, then p_1 is not conserved, but p_2 and p_3 are.

$L \neq L\,(x^i)$ means p_i conserved

Note this means the Lagrangian is symmetric. As we saw on pg. 167, whenever a function is not dependent on a coordinate, it is symmetric in that coordinate. Changing the coordinate via a translation to a new coordinate value makes no change in the function.

Bottom line: If the Lagrangian is symmetric in a coordinate, then the conjugate momentum for that coordinate is conserved. This is true not just for the coordinates x^i, but for any generalized coordinates q^i (We reviewed generalized coordinates in Chap. 2. $q^i = x^i$ is just a special case).

L symmetric in generalized coord q^i means conjugate momentum p_i conserved

This should be review for you, but if you feel you need more practice with symmetry of L and conservation, do Probs. 10 and 11.

Symmetry of the Lagrangian Density

The prior section dealt with the Lagrangian of a particle, and similar effects arise for the Lagrangian density $\mathcal{L}$ of classical field theory. We will not, however, delve into that here, but simply move on to symmetries of $\mathcal{L}$ in quantum field theory and what they imply there.

Similar effect with Lagrangian density

6.3 Transformations in Quantum Field Theory

6.3.1 Scalars, Vectors, and Tensors

We have discussed, in the classical mechanics section above, how scalars and vectors transform under the Lorentz (and Poincare') transformation. The same conclusions carry over to the fields of QFT. However, there are no spinors in classical theory, so we didn't discuss their transformation properties there, but we need to in QFT.

Quantum scalars, vectors transform in same way as classical

6.3.2 Spinor Transformations

When we transform our coordinate system either by a Lorentz transformation (boost = change in velocity) or a rotation of our coordinates (change in angle), we ask how a spinor field will transform. We know world scalars maintain their same value at an event and vectors transform according to (6-14). For spinors, we seek a matrix which is four by four in spinor space and which represents what happens to a spinor under a Lorentz transformation and/or a rotation of coordinates. That is, we seek D in

Spinors only in QM

$$\psi'\left(x'^{\mu}\right)=D\psi\left(x'^{\mu}\right)=D\psi\left(\Lambda^{\mu}{}_{\nu}x^{\nu}\right)\xrightarrow[\text{written out}]{\text{with spinor indices}}\psi'_{\alpha}\left(x'^{\mu}\right)=D_{\alpha\beta}\psi_{\beta}\left(x'^{\mu}\right). \tag{6-21}$$

Spinors have their own transformation

Deriving D can take pages, is quite complicated, and would take us far afield from our present direction, so I will just state it. Interested readers can find this derivation in certain texts or online[3]. The spinor transformation under Lorentz and rotation transformations is

$$D=e^{-i(\mathbf{L}\cdot\boldsymbol{\Theta}+\mathbf{M}\cdot\mathbf{Q})}\qquad L^{k}=-\tfrac{i}{2}\varepsilon_{ij}{}^{k}\gamma^{i}\gamma^{j},\ \ \Theta^{k}=\left(\theta^{1},\theta^{2},\theta^{3}\right),\ \ M^{k}=\tfrac{i}{2}\gamma^{0}\gamma^{k},\ \ Q^{k}=\left(v^{1},v^{2},v^{3}\right), \tag{6-22}$$

where Θ^{k} represents rotation angles of the primed system with respect to the unprimed system; Q^{k} is a three vector of the boost velocities; and $\varepsilon_{ij}{}^{k}$ is zero unless i,j,k are all different, 1 if their order is 1,2,3 or 2,3,1 or 3,1,2, and – 1 for other orders.

It is probably not beneficial, at this point, to worry too much more about (6-22). If, at some time in the future, your work takes you in a direction where you need to understand this transformation better, then you can study it more extensively then.

Note that in formal mathematical language, the set of all possible Lorentz transformations (all possible **v**) is known as the Lorentz group. When the Lorentz group acts on the coordinate system, it changes what our spinors look like in the new system and this change is represented by D. So D is called a representation of the Lorentz group. It "represents" that group in spinor space.

Spinor transformation is a representation of the Lorentz group

We note, again without proof here due to complexities involved, that

$$\bar{\psi}\psi=\text{world scalar}\qquad\bar{\psi}\gamma^{\mu}\psi=\text{transforms like four vector}\,. \tag{6-23}$$

Spinor objects that transform like world scalars and vectors

The first part of Wholeness Chart 6-3 summarizes Lorentz transformations for scalars, vectors, spinors and tensors (for those experienced with tensors).

6.4 Lorentz Symmetry of the Lagrangian Density

As reviewed in Chap. 2, XXX Sect. 2.5.1, point 11, pg. 25 XXX, the Lagrangian density is a Lorentz scalar, in the sense that it has the same value at any event (4D point) as seen from any inertial coordinate system. But there is a deeper symmetry that $\mathcal{L}$ has as well. Its functional form, in terms of the fields of which it is composed is also the same in any inertial coordinate system. It is a symmetric *function*, in addition to being a symmetric *value* at every (4D) point.

$\mathcal{L}$ symmetric under Lorentz transformation

We conclude this because of Einstein's postulate that the laws of nature (the field equation here) are invariant in form under Lorentz transformation. The Euler-Lagrange equation for fields, which is another form of the field equation, is a law of nature and must therefore be invariant in form as well. But since $\mathcal{L}$ is inserted into the invariant Euler-Lagrange equation to get the invariant field equation, $\mathcal{L}$ itself must be invariant in functional form. $\mathcal{L}$ is the same function of the fields in any inertial coordinate system.

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[3] For example, David Tong's lecture http://www.damtp.cam.ac.uk/user/dt281/qft/four.pdf , or Sidney Coleman's notes www.quantumfieldtheory.info/Sydney_Coleman_QFT_lecture_notes.pdf pg. 125.

This is summarized, with a concrete example from scalar field theory, in the last three rows of Wholeness Chart 6-3 below.

Note that, as discussed in the referenced page above, and the associated problem thereto, though $\mathcal{L}$ is a world scalar, L is not. Neither is the Hamiltonian H nor the Hamiltonian density $\mathcal{H}$. That is, none of L, H, and $\mathcal{H}$ have the same value when measured in coordinate systems having relative velocity to one another.

Wholeness Chart 6-3. Summary of Effect of Lorentz Transformation on Fields

	x^μ **system**	x'^μ **system**	**Comment**
	$x^\mu \longrightarrow$ Lorentz transformation $\Lambda \longrightarrow x'^\mu$		symbols defined: $x'^\alpha = \Lambda x^\alpha = \Lambda^{\alpha'}{}_\alpha x^\alpha$
Scalar field	$S(x^\mu)$	$S'(x'^\mu) = S(x^\mu)$ always $S(x'^\mu) = S(x^\mu)$ if sym form	x'^μ and x^μ represent same event $\leftarrow$ If S is symmetric function under Λ, i.e., same function of x'^μ as of x^μ
Vector field	$V^\mu(x^\alpha)$	$V'^\mu(x'^\alpha) = \Lambda V^\mu(\Lambda x^\alpha)$ always $V'^\mu(x'^\alpha) = V^\mu(x^\alpha)$ if sym	$V'^\mu V'_\mu = V^\mu V_\mu$ invariant, V^μ covariant 2 vectors, $V'^\mu W'_\mu = V^\mu W_\mu$ invariant $\leftarrow$ If V^μ components sym under Λ
Tensor field	$T^{\mu\nu}(x^\alpha)$	$T'^{\mu\nu}(x'^\alpha) = \Lambda\Lambda T^{\mu\nu}(\Lambda x^\alpha)$ $T'^{\mu\nu}(x'^\alpha) = T^{\mu\nu}(x^\alpha)$ if sym	$T'^{\mu\nu}T'_{\mu\nu} = T^{\mu\nu}T_{\mu\nu}$ invar, $T^{\mu\nu}$ covar Other invariants exist such as trace $T^\mu{}_\mu$ $\leftarrow$ If $T^{\mu\nu}$ components sym under Λ
Spinor field	$\psi(x^\alpha)$	$\psi'(x'^\alpha) = D\psi(\Lambda x^\alpha)$ $\bar{\psi}\psi$ invariant $\left(\bar{\psi}\gamma^\mu\psi\right)' = \Lambda^\mu{}_\nu \bar{\psi}\gamma^\nu\psi$	D = Lorentz group rep for spinors $\bar{\psi}\psi$ transforms like world scalar $\bar{\psi}\gamma^\mu\psi$ transforms like 4-vector
Law of nature	$(\partial^\mu\partial_\mu + m)\,\phi(x^\alpha) = 0$	$(\partial'^\mu\partial'_\mu + m)\,\phi'(x'^\alpha) = 0$	Same form under Λ Example is Klein-Gordon field equation
Euler-Lagrange equation	$\dfrac{\partial}{\partial x^\mu}\left(\dfrac{\partial\mathcal{L}}{\partial\phi^r{}_{,\mu}}\right) - \dfrac{\partial\mathcal{L}}{\partial\phi^r} = 0$	$\dfrac{\partial}{\partial x'^\mu}\left(\dfrac{\partial\mathcal{L}'}{\partial\phi'^r{}_{,\mu}}\right) - \dfrac{\partial\mathcal{L}'}{\partial\phi'^r} = 0$	Same form under Λ
Lagrangian density	$\mathcal{L}(\phi^r(x^\alpha))$ $\mathcal{L} = \partial_\alpha\phi^\dagger\partial^\alpha\phi - \mu^2\phi^\dagger\phi$	$\mathcal{L}'(\phi'^r(x'^\alpha)) = \mathcal{L}(\phi'^r(x'^\alpha))$ $\mathcal{L} = \partial_\alpha\phi'^\dagger\partial^\alpha\phi' - \mu^2\phi'^\dagger\phi'$	$\mathcal{L}$ into above $\rightarrow$ law of nature, so $\mathcal{L}$ must have same form under Λ, as well Example is Klein-Gordon $\mathcal{L}$

6.5 Other Symmetries of the Lagrangian Density: Noether's Theorem

6.5.1 Example of a Different Kind of Symmetry

There are other ways the Lagrangian density can be symmetric, other than under Lorentz transformations. For example, consider the scalar Lagrangian (density)

A different kind of symmetry for $\mathcal{L}$

$$\mathcal{L}_0^0 = \left(\partial_\alpha \phi^\dagger \partial^\alpha \phi - \mu^2 \phi^\dagger \phi\right) \tag{6-24}$$

where we introduce a transformation that changes the phase angle of the solution

$$\phi \rightarrow \phi' = \phi \, e^{-i\alpha} \tag{6-25}$$

Under this transformation, $\mathcal{L}$ is symmetric

where α is a real constant. No change is made to x^μ of which ϕ is a function.

What does such a transformation do to the Lagrangian? (We will start dropping the word "density", as is common practice) Note from (6-25) that $\phi = \phi' \, e^{i\alpha}$, and plugging that into (6-24) we have

$$\mathcal{L}_0^0\left(\phi^\dagger, \phi\right) = \left(\partial_\alpha \phi^\dagger \partial^\alpha \phi - \mu^2 \phi^\dagger \phi\right) \overset{\phi \rightarrow \phi' = \phi \, e^{-i\alpha}}{\rightarrow} \mathcal{L}_0^0 = \left(\partial_\alpha \underbrace{\phi'^\dagger e^{-i\alpha}}_{\phi^\dagger} \partial^\alpha \underbrace{\phi' e^{i\alpha}}_{\phi} - \mu^2 \underbrace{\phi'^\dagger e^{-i\alpha}}_{\phi^\dagger} \underbrace{\phi' e^{i\alpha}}_{\phi}\right) \tag{6-26}$$

$$= \left(\partial_\alpha \phi'^\dagger \partial^\alpha \phi' - \mu^2 \phi'^\dagger \phi'\right) = \mathcal{L}_0^0\left(\phi'^\dagger, \phi'\right).$$

So the Lagrangian is unchanged in form under the transformation. The transformed Lagrangian has the same form whether in terms of ϕ or ϕ'. Thus, the law of nature derived from the Lagrangian, the Klein-Gordon equation in this case, also looks the same in terms of ϕ or ϕ'. And so, all predictions for measurements using either solution will be the same.

6.5.2 Internal vs External Symmetries

External vs internal symmetries

Poincare' transformations (Lorentz plus 4D translation) and 3D rotations involve changes to our physical coordinates x^μ of our external world and are called external transformations.

Transformations like (6-25) have nothing to do with x^μ, but instead function in hidden spaces, behind the scene, like Hilbert or Fock space. They are called internal transformations.

In both cases, if something remains the same under the transformation, we have a symmetry (external or internal.)

Internal transformations like rotations and reflections in abstract space

Note that the transformation (6-25) amounts to a rotation in the complex plane, which is an internal space. We will see repeatedly, as we delve further into QFT, that internal transformations often amount to what can be visualized in some cases as akin to rotations, and in others, as reflections, in non-physical, abstract spaces like Fock space.

6.5.3 Noether's Theorem

A general theorem to cover all types of transformations, but most useful for internal transformations, was discovered by Emily Noether and bears her name. It plays an extremely important role in QFT, and in words, can be stated like this.

Noether's theorem in words

Noether's theorem (without math): A symmetry in the Lagrangian density implies an associated quantity is conserved.

This is reminiscent of the symmetry in the Lagrangian L of classical particle theory with respect to the generalized coordinate q^i (x^i in a Cartesian system) (see pg. 171) implying the conjugate momentum p_i is conserved.

Proof of Noether's Theorem:

Proof of Noether's theorem

If $\mathcal{L}$ is symmetric in some parameter α, then it is unchanged when α changes, i.e., its derivative with respect to α is zero.

$$\mathcal{L} = \mathcal{L}\left(\phi^r, \phi^r{}_{,\mu}\right) \text{ symmetric in } \alpha, \text{ then } \quad \rightarrow \quad \frac{\partial \mathcal{L}}{\partial \alpha} = 0 = \underbrace{\frac{\partial \mathcal{L}}{\partial \phi^r}}_{\substack{\text{use Euler-}\\ \text{Lagrange}\\ \text{equation}}} \frac{\partial \phi^r}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial \phi^r{}_{,\mu}} \frac{\partial \phi^r{}_{,\mu}}{\partial \alpha} \tag{6-27}$$

Using $\partial \mathcal{L} / \partial \phi^r$ from the Euler-Lagrange field equation

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\right)-\frac{\partial \mathcal{L}}{\partial \phi^{r}}=0 \tag{6-28}$$

in (6-27), yields

$$0=\left(\frac{\partial}{\partial x^{\mu}}\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\right)\frac{\partial \phi^{r}}{\partial \alpha}+\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\frac{\partial \phi^{r}{}_{,\mu}}{\partial \alpha}=\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\frac{\partial \phi^{r}}{\partial \alpha}\right)-\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\underbrace{\frac{\partial}{\partial x^{\mu}}\frac{\partial \phi^{r}}{\partial \alpha}}_{\frac{\partial \phi^{r}{}_{,\mu}}{\partial \alpha}}+\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\frac{\partial \phi^{r}{}_{,\mu}}{\partial \alpha}\,. \tag{6-29}$$

The last two terms cancel, leaving

$$\frac{\partial}{\partial x^{\mu}}\underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\frac{\partial \phi^{r}}{\partial \alpha}\right)}_{j^{\mu}}=0 \quad\rightarrow\quad \partial_{\mu} j^{\mu}=0 \quad\rightarrow\quad \int_{\substack{\text{all}\\\text{space}}} j^{0} d^{3}x=Q'=\text{constant in time}\,. \tag{6-30}$$

The first two expressions above are simply our old friend the continuity equation for the quantity j^{μ}. And that means the zeroth component of j^{μ} is a density value that when integrated over all space is conserved.

End of proof.

Noether's theorem (mathematically): If the Lagrangian density $\mathcal{L}(\phi^{r}, \phi^{r}{}_{,\mu})$ is symmetric in form with respect to a transformation in ϕ^{r} which is a function of a parameter α, i.e., $\phi^{r}(x^{\mu}) \rightarrow \phi^{r}(x^{\mu},\alpha)$, then the four current (using $\phi^{r}(x^{\mu},\alpha)$)

$$j^{\mu}\left(\phi^{r},\phi^{r}{}_{,\mu}\right)=\frac{\partial \mathcal{L}}{\partial \phi^{r}{}_{,\mu}}\frac{\partial \phi^{r}}{\partial \alpha} \quad (\text{sum on } r) \tag{6-31}$$

has zero four-divergence, $\partial_{\mu} j^{\mu}=0$. Thus its zeroth component j^{0} integrated over all space is conserved, as is qj^{0} integrated over all space, where q is a constant.

Noether's theorem mathematically

6.5.4 Applying to Our Example

Let's use (6-31) with our example of Sect. 6.5.1 above. Our symmetry transformation is (6-25). That is, we showed there that the scalar Lagrangian (6-24) is invariant in form under (6-25). But now we want to know what exactly is conserved under this symmetry.

Applying Noether's theorem

From (6-24), we find the terms for the first factor on the RHS of (6-31) (note that summation over r in (6-31) has r = 1 for ϕ and r = 2 for $\phi^{\dagger}$)

$$\begin{aligned}\frac{\partial \mathcal{L}_{0}^{0}}{\partial \phi_{,\mu}}&=\frac{\partial}{\partial \phi_{,\mu}}\left(\phi^{\dagger}{}_{,\alpha}\phi^{,\alpha}-\mu^{2}\phi^{\dagger}\phi\right)=\frac{\partial}{\partial \phi_{,\mu}}\left(\phi^{\dagger}{}_{,\alpha}\phi^{,\alpha}\right)=\underbrace{\frac{\partial \phi^{\dagger}{}_{,\alpha}}{\partial \phi_{,\mu}}}_{0}\phi^{,\alpha}+\phi^{\dagger}{}_{,\alpha}\frac{\partial \phi^{,\alpha}}{\partial \phi_{,\mu}}=\phi^{\dagger}{}_{,\alpha}g^{\alpha\mu}=\phi^{\dagger,\mu}\\ \frac{\partial \mathcal{L}_{0}^{0}}{\partial \phi^{\dagger}{}_{,\mu}}&=\frac{\partial}{\partial \phi^{\dagger}{}_{,\mu}}\left(\phi^{\dagger}{}_{,\alpha}\phi^{,\alpha}-\mu^{2}\phi^{\dagger}\phi\right)=\frac{\partial}{\partial \phi^{\dagger}{}_{,\mu}}\left(\phi^{\dagger}{}_{,\alpha}\phi^{,\alpha}\right)=\frac{\partial \phi^{\dagger}{}_{,\alpha}}{\partial \phi^{\dagger}{}_{,\mu}}\phi^{,\alpha}+\underbrace{\frac{\partial \phi^{,\alpha}}{\partial \phi^{\dagger}{}_{,\mu}}}_{0}\phi^{\dagger}{}_{,\alpha}=\delta_{\alpha}^{\mu}\phi^{,\alpha}=\phi^{,\mu}\,.\end{aligned} \tag{6-32}$$

We find the second factor on the RHS of (6-31) from the transformation relation (6-25) $\phi \rightarrow \phi e^{i\alpha}$,

$$\begin{aligned}\frac{\partial \phi(x^{\mu},\alpha)}{\partial \alpha}&=\frac{\partial}{\partial \alpha}\phi(x^{\mu})\,e^{-i\alpha}=-i\phi(x^{\mu})\,e^{-i\alpha}\\ \frac{\partial \phi^{\dagger}(x^{\mu},\alpha)}{\partial \alpha}&=\frac{\partial}{\partial \alpha}\phi^{\dagger}(x^{\mu})\,e^{i\alpha}=i\phi^{\dagger}(x^{\mu})\,e^{i\alpha}\,.\end{aligned} \tag{6-33}$$

Using (6-32) and (6-33) in (6-31) we find

$$j^{\mu}\left(\phi^{r},\phi^{r}{}_{,\mu}\right)=\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\frac{\partial\phi}{\partial\alpha}+\frac{\partial\mathcal{L}}{\partial\phi^{\dagger}{}_{,\mu}}\frac{\partial\phi^{\dagger}}{\partial\alpha}=-i\underbrace{\phi^{\dagger,\mu}(x^{\mu},\alpha)}_{\phi^{\dagger,\mu}(x^{\mu})\,e^{i\alpha}}\phi(x^{\mu})\,e^{-i\alpha}+i\underbrace{\phi^{,\mu}(x^{\mu},\alpha)}_{\phi^{,\mu}(x^{\mu})\,e^{-i\alpha}}\phi^{\dagger}(x^{\mu})\,e^{i\alpha}$$
$$=i\left(\phi^{,\mu}(x^{\mu})\phi^{\dagger}(x^{\mu})-\phi^{\dagger,\mu}(x^{\mu})\phi(x^{\mu})\right). \tag{6-34}$$

We get same four current as we found for charge in Chap 3

This is identical to the scalar four-current, with zero four-divergence and conserved Q' (see (6-30) of Chap. 3 XXX (3-23), pg. 46 XXX. In Chap. 3, we found this using the Klein-Gordon equation. Here, we found it using Noether's theorem. (Richard Feynman once said that a good physicist should be able to find the same result via different paths.)

From the ensuing discussion in Chap. 3, we learned that in RQM, j^0 can be interpreted for a particle as probability density and qj^0, where q is charge on a single particle, as charge density. So in RQM, $\int q\, j^0 d^3x=\int s^0 d^3x=q$ is conserved, as charge must be. Obviously, if $\partial_{\mu} j^{\mu}=0$, then so does $\partial_{\mu}\left(q\, j^{\mu}\right)=\partial_{\mu}s^{\mu}=0$.

6.5.5 Charge Operator in QFT

As we learned in Chaps. 3, 4, and 5, in QFT entities like j^{μ} are a little different in the sense that they are really operators that operate on states. Indeed, if we follow the steps we did in Chap. 3 in exactly the same way (use (6-34) to find j^{0}, plug that into the RHS of (6-30), and multiply by an arbitrary constant q equal to the charge on one particle), we find

$$Q=qQ'=q\int j^0 d^3x=\int s^0 d^3x=q\sum_{\mathbf{k}}\left(N_a\left(\mathbf{k}\right)-N_b\left(\mathbf{k}\right)\right). \tag{6-35}$$

What is really physically conserved is the charge of the multiparticle state, i.e.,

$$Q\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle=q\sum_{\mathbf{k}}\left(N_a\left(\mathbf{k}\right)-N_b\left(\mathbf{k}\right)\right)\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle$$
$$=\underbrace{q\left(n_1+n_2+..-\bar{n}_1-..\right)}_{\text{conserved}}\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle. \tag{6-36}$$

When we say an operator is conserved, we really mean the associated physical value is

Above, we simply stated the measured charge is conserved. To prove it, consider the following.

Proof that "conservation of an operator" derivation means conserved measured quantity

The state (ket) in (6-36) is an eigenstate of the charge operator Q, and in fact, every state with a given number of particles is a charge eigenstate. That is, if we measure the total charge of a given multiparticle state, we will get a certain number. If we imagined we had an exact duplicate of that multiparticle state at the same moment in time, and then measured its charge, we would get the same number again. Repeating this duplication and measurement, we would always get the same number eigenvalue for total charge. That, of course, is the characteristic of an eigenstate. A general state superposition of eigenstates would sometimes measure one eigenvalue and sometimes another.

Proof that conserved operator means physical value is conserved.

The average of (imagined) repeated measurements of the same state at the same moment in time is, of course, the expectation value of the quantity measured. For an eigenstate, then, this average, the expectation value, is the same as the eigenvalue that is measured each time.

$$\bar{Q}=\left\langle n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right|Q\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle$$
$$=\left\langle n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right|q\sum_{\mathbf{k}}\left(N_a\left(\mathbf{k}\right)-N_b\left(\mathbf{k}\right)\right)\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle$$
$$=\left\langle n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right|q\left(n_1+n_2+..-\bar{n}_1-..\right)\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle \tag{6-37}$$
$$=q\left(n_1+n_2+..-\bar{n}_1-..\right)\underbrace{\left\langle n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\middle|\middle|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle}_{=1}=q\left(n_1+n_2+..-\bar{n}_1-..\right).$$

So, if we ask, how the expectation value of an eigenstate changes over time, we are asking how the measured eigenvalue changes over time. We are asking if the time derivative of (6-37) is zero. If it is, $\bar{Q}$ is conserved. Thus,

$$\frac{d\bar{Q}}{dt}=\underbrace{\left(\frac{d}{dt}\left\langle n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right|\right)}_{=0\text{ in Heisenberg picture}}Q\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle$$

$$+\left\langle n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right|\underbrace{\frac{dQ}{dt}}_{=0}\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle \qquad (6\text{-}38)$$

$$+\left\langle n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right|Q\underbrace{\left(\frac{d}{dt}\left|n_1\phi_1,n_2\phi_2,...\bar{n}_1\bar{\phi}_1,...\right\rangle\right)}_{=0\text{ in Heisenberg picture}}=0.$$

Saying an operator is conserved means its expectation value is

Note from Chap. 2 XXX Wholeness Chart 2-4, pg. 28 XXX, that states do not have time dependence in the Heisenberg picture, the picture that we employ for QFT free fields. The middle line above is zero because we showed in (6-30) that the operator Q is conserved ($Q = qQ'$), and thus its time derivative is zero. So, the total time derivative of $\bar{Q}$ is zero. (The same conclusion would be reached in the Schrödinger picture, but it would be a little more complicated to derive.)

End proof

For eigenstate of an operator, expectation value = eigenvalue

Bottom line: The expectation value (expected measurement) of a conserved operator is conserved. If the state measured is in an eigenstate, any measurement at any time will yield the same eigenvalue.

So eigenvalue is conserved if operator is "conserved"

So when we cavalierly say in QFT that Q is conserved, remember that Q is really an operator, which it is difficult to think of as being conserved, and that the real thing conserved is the numeric result of operating on a ket with Q. Keep in mind, however, that in QFT, virtually everyone speaks of the operator itself as being conserved.

As the particles represented by the ket of (6-36) move through the universe, each time we operate on that ket with Q, we will get the same number eigenvalues, the same charge.

Similarly, j^μ is also an operator and its zero four-divergence really means that the corresponding component numeric values for the physical particles represented by the ket it would act on, have zero four-divergence. For example, operation of $\partial_\mu j^\mu = 0$ on that ket would always yield zero times the ket.

We have been dealing strictly with free particles, but we will soon find, and Noether's theorem will help us to do it, that interacting particles conserve total charge as well. This is something we know already is true in the physical world, of course, but our theory would hardly be worth anything if we didn't find the same thing there.

6.5.6 More on Symmetry and Noether's Theorem

Spinors and Vectors

Spinors and vectors similar to above example for scalars

It should come as little surprise that spinor and vector four currents, giving rise to conserved charge, such as we found in Chaps. 4 and 5, can be derived from Noether's theorem, as well. You can prove that to yourself, if you really need to, by doing Probs. 12 and 13.

Other test for conservation: commuting with Hamiltonian

Does $[H,Q]=0$ mean conservation of Q, as it did in NRQM?

You may recall that in NRQM, a dynamical variable was conserved if its operator commuted with the Hamiltonian. That is, for an operator $\mathcal{O}$ with corresponding dynamical variable numeric value $\mathcal{O}$

$$[H,\mathcal{O}]=0 \qquad \text{means} \qquad \frac{d\mathcal{O}}{dt}=0. \qquad (6\text{-}39)$$

You may, at some point, have wondered why this wasn't used in the development of QFT. It is a good question. So, does this test for conservation hold in QFT as well?

To answer, consider our scalar charge operator (6-35) and the scalar Hamiltonian operator from Chap. 3 expressed in terms of number operators. Number operators commute, for example,

$$N_a(\mathbf{k})N_b(\mathbf{k}')\left|n_{\mathbf{k}}\phi_{\mathbf{k}},\bar{n}_{\mathbf{k}'}\bar{\phi}_{\mathbf{k}'}\right\rangle = N_a(\mathbf{k})\bar{n}_{\mathbf{k}'}\left|n_{\mathbf{k}}\phi_{\mathbf{k}},\bar{n}_{\mathbf{k}'}\bar{\phi}_{\mathbf{k}'}\right\rangle = n_{\mathbf{k}}\bar{n}_{\mathbf{k}'}\left|n_{\mathbf{k}}\phi_{\mathbf{k}},\bar{n}_{\mathbf{k}'}\bar{\phi}_{\mathbf{k}'}\right\rangle$$
$$= \bar{n}_{\mathbf{k}'}n_{\mathbf{k}}\left|n_{\mathbf{k}}\phi_{\mathbf{k}},\bar{n}_{\mathbf{k}'}\bar{\phi}_{\mathbf{k}'}\right\rangle = N_b(\mathbf{k}')N_a(\mathbf{k})\left|n_{\mathbf{k}}\phi_{\mathbf{k}},\bar{n}_{\mathbf{k}'}\bar{\phi}_{\mathbf{k}'}\right\rangle. \quad (6\text{-}40)$$

Number operators commute

So

$$[H,Q] = \left[\sum_{\mathbf{k}}\mathbf{k}\left(N_a(\mathbf{k}) - N_b(\mathbf{k})\right), q\sum_{\mathbf{k}}\left(N_a(\mathbf{k}) - N_b(\mathbf{k})\right)\right] = 0 \quad (6\text{-}41)$$

H and Q commute. And charge q is conserved. We conclude that this means for determining whether or not a quantity is conserved is valid in QFT, as it was in NRQM.

Yes, $[H,Q]=0$ *means conservation of Q in QFT, too.*

We caution, however, that no one (at least in my experience) in QFT ever uses (6-39) to do so, and no text I know of shows it. Noether's theorem comprises the canon in that regard. But since one so often has the experience in learning QFT of wondering where some basic principle of NRQM went to in this new and very different theory, I felt it good to provide this discussion of it.

But $[H,Q]=0$ *almost never used in QFT*

The various ways to determine a conserved quantity are listed in Wholeness Chart 6-4.

Other uses

Noether's theorem is used repeatedly throughout QFT, and we will eventually see it can tell us whether weak and strong charges are conserved, as well. It also can be used to determine that total energy and momentum are conserved. (See Probs. 14 and 15.) There are still other uses for symmetry as we will see when we get to interactions.

Noether's theorem has wide range of applications

Wholeness Chart 6-4. Ways to Determine If a Quantity is Conserved

	1st Method	**2nd Method**	**3rd Method**
Steps	Manipulating wave equation and its complex conjugate	Noether's theorem	Operator Q' commutation with H
Result	Four current with zero divergence, $\partial_\mu j^\mu = 0$	Four current with zero divergence, $\partial_\mu j^\mu = 0$	$[Q',H] = 0$
Meaning	$\int j^0 d^3x = Q'$ conserved	$\int j^0 d^3x = Q'$ conserved	Q' conserved
Application	$Q = qQ'$ = electric charge (conserved)		
Other applications	Could be used for weak and strong charge conservation, but not common	Weak and strong charge conservation, energy and 3 momentum conservation	As at left.

6.6 Symmetry, Gauges, and Gauge Theory

6.6.1 A Simple Example and Definitions

You have probably heard quantum electromagnetic, weak, and strong force theories called gauge theories. So are other theories you are already familiar with, such as the classical gravitational and electromagnetic field theories. See Chap. 5, XXX pgs 138-142 XXX. As a very simple example, consider an electrostatic field potential $\Phi(\mathbf{x})$ where **E,** the force on a particle per unit charge q, is

Simple example of a classical gauge field transformation

$$\mathbf{E} = -\nabla\Phi = -\nabla\left(\Phi + C\right) = -\nabla\Phi' \qquad \Phi' = \Phi + C \qquad C = \text{constant}\,. \quad (6\text{-}42)$$

Our measurable **E** is the same for Φ or Φ'. So **E** is symmetric under the transformation $\Phi \rightarrow \Phi'$. We call Φ (or Φ') our gauge field; and $\Phi \rightarrow \Phi' = \Phi + C$, the gauge transformation. Each different configuration Φ' is a different gauge of the gauge field. That is, for each different value of C, we have a different gauge (for the same gauge field.)

In Chap. 5 this got more complicated for electrodynamics, where we also had a vector potential (gauge vector field) **A**.

Definitions

Gauge invariance (or gauge symmetry) is the property of a field theory in which different configurations of the underlying fundamental, but unobservable, field(s) result in identical observable properties.

Definitions related to gauge theory

The unobservable field, often a potential field, is called the gauge field.

A gauge transformation changes the gauge field from one configuration to another.

Each different configuration of the gauge field is a different gauge.

A theory having gauge invariance (symmetry) is called a gauge theory.

6.6.2 Free Quantum Field Theory and Gauges

Recall from Wholeness Chart XXX 3-4 XXX at the end of Chap. 3 that the fields such as ϕ, ψ, and A^μ are themselves not observable. They cannot be measured directly. (We prove that they have zero expectation value in Chap. 7.) But properties of the fields like energy, momentum, and charge are measurable. Our dynamical variable operators, which include number operators, reflect this. They typically have non-zero expectation values.

Quantum gauge theories

Note that under the transformation (6-25), repeated below,

Scalar gauge transformation

$$\phi \rightarrow \phi' = \phi e^{-i\alpha} , \qquad (6\text{-}43)$$

the Lagrangian (6-26) remained invariant. Thus, the Klein-Gordon field equation derived from that Lagrangian is invariant, i.e., ϕ' solves the K-G equation as well as ϕ. All our dynamical variable operators are ultimately derived from the Lagrangian, so they too will be the same for ϕ'. As one example, see the 4-current of (6-34) in which we effectively substitute ϕ' for ϕ and get the same result for j^μ and Q.

Free QFT is a gauge theory, because $\mathcal{L}$, and thus measurables, unchanged under gauge transformation

The transformation (6-43) is a gauge transformation of the underlying, unobservable field ϕ. The theory of free scalar quantum fields is a gauge theory, because all measurable quantities are unchanged under the gauge transformation. By doing Prob. 16, you can show the same thing is true for free Dirac spinor field fields.

Note that the gauge transformation (6-43) is simply a change in phase of the field. This is similar to NRQM, where we may recall that we could change the phase of the wave function, but observables like probability density, energy, and momentum remained unchanged. A solution to the Schrödinger equation could have any constant phase factor and still be a solution predicting the same measurable results.

Transformation here is a phase change Observables unchanged like phase change in NRQM

Thus, gauge symmetries are internal symmetries. (See Sect. 6.5.2, pg. 174.)

Gauge symmetry is an internal symmetry

More formal definition

We can also say that a gauge theory is a type of field theory in which the Lagrangian (density) is invariant under a continuous (not discrete) transformation.

6.7 Chapter Summary

We have seen that for

Symmetry and transformations in general

- symmetry is the propensity for non-change with superficial change
- mathematically, symmetry is invariance under transformation
- Wholeness Chart 6-1 compares and contrasts symmetric and non-symmetric transformations

Scalar value at a point is always invariant. Scalar function form invariant only under symmetry transformation

Vector components at a point vary co-variantly. Vector length and direction in physical space same under any transformation. Vector function form invariant only under symmetry transformation

A scalar or vector function that is not a function of a coordinate x^j is symmetric with respect to a displacement in the j coordinate direction.

Transformations in classical mechanics

- the laws of nature are symmetric under Lorentz transformation, i.e., invariant in spacetime
- symmetry of the classical Lagrangian L under a translation transformation of a generalized coordinate q^j (often x^j) means the conjugate momentum p_j of that coordinate is conserved
- similar effects for classical Lagrangian density $\mathcal{L}$

Transformations in QFT (see Wholeness Chart 6-3)

- scalar and vector quantum fields transform like classical ones did; spinors do not exist classically, but have their own form for QFT transformations
- symmetry of the QFT Lagrangian density $\mathcal{L}$ under Lorentz transformation means field equation (law of nature) is invariant in form for different observers
- Noether's theorem: If $\mathcal{L}$ is symmetric under a change of a parameter, then there is an associated quantity that is conserved
- there are three ways to determine if a quantity is conserved (see Wholeness Chart 6-4), though Noether's theorem method is the most useful and covers widest range of cases.
- a gauge theory is a field theory for which the Lagrangian (and thus all measurables) remains invariant under a transformation of the underlying unmeasurable gauge field
- a gauge symmetry is an internal symmetry; a Lorentz symmetry is an external symmetry

6.8 Problems

1. Is the function $F = 2(x^1)^2 + (x^2)^2$ symmetric under rotation in the x^1-x^2 plane? Guess first, then prove (or disprove) your answer by expressing F in terms of a rotated set of coordinates x'^1-x'^2, i.e, as $F'(x'^1, x'^2)$, where θ is the angle of rotation between the two coordinate systems.

2. In Prob. 1, at the point $(x^1, x^2) = (1,2)$, F has the value 6. If we transform to the rotated coordinate system x'^1-x'^2 with $\theta = 45^o$, what are the coordinates of that same physical point in space in that coordinate system? Using your expression F' for F in terms of x'^1 and x'^2, show that $F'(x'^1, x'^2)$ at that physical point equals 6, as well.

3. Without doing any calculations, is the function $G = (x^1)^2 + (x^2)^2 + (x^3)^2$ symmetric under rotation in 3D space? Is $H = (x^1)^2 + 3(x^2)^2 + (x^3)^2$?

4. Is the function $J = (x^1)^2 + (x^3)^2$ symmetric under the translation $x^2 \rightarrow x'^2 = x^2 + a$, where a is a constant? Is it symmetric under $x^3 \rightarrow x'^3 = x^3 + a$?

5. Is the differential equation $\partial_i x^i = 3$ symmetric under the translation $x^2 \rightarrow x'^2 = x^2 + a$, where a is a constant? Is it symmetric under $x^2 \rightarrow x'^2 = x^2 + (x^2)^2$?

6. Consider the position vector $(x^1, x^2) = (3,4)$. This vector's length is 5, and for the x^1 axis horizontal, its angle with the horizontal is 53^o. What are this vector's position coordinates in the x'^1-x'^2 coordinate system of Prob. 1? What is its length? Calculate it. What is its angle with the horizontal? What is its angle with respect to the x'^1 axis? Express your answer in terms of θ.

7. On page 167 we briefly discussed the spherical symmetry of the electric field around a point charge. It is easier mathematically to consider the symmetry of the simpler case of an infinitely long line of uniformly distributed charge. This radiates an electric field **E** in a coordinate system with x^3 axis aligned with the line of charge of components (where ϕ below is the relevant cylindrical coordinate system angle)

$$\begin{bmatrix} E^1 \\ E^2 \\ E^3 \end{bmatrix} = \frac{E_0}{r}\begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} = \frac{E_0}{r}\begin{bmatrix} x^1/r \\ x^2/r \\ 0 \end{bmatrix} = \frac{E_0}{\left(x^1\right)^2 + \left(x^2\right)^2}\begin{bmatrix} x^1 \\ x^2 \\ 0 \end{bmatrix}.$$

Express E^i and x^i above in the primed coordinate system of Fig. 6-2 on page 165 using (6-3) to show that

$$\begin{bmatrix} E'^1 \\ E'^2 \\ E'^3 \end{bmatrix} = \frac{E_0}{\left(x'^1\right)^2 + \left(x'^2\right)^2} \begin{bmatrix} x'^1 \\ x'^2 \\ 0 \end{bmatrix},$$

and thus, that the vector field components E^i in this case are symmetric under the rotation transformation of Fig. 6-2. If you feel ambitious, repeat the analysis for the **E** field around a point charge.

8. Show that $F^i = m\ddot{x}^i$ is symmetric under the transformation $x^i \to x^i + a^i$, where a^i is a constant for each i.

9. Transform the components in (6-15) by the Lorentz transformation (6-13) and show that $w_\mu w^\mu = w'_\mu w'^\mu$.

10. For a particle attached to a spring confined to move in one dimension, the potential energy $V = ½\, k\, (x^1)^2$. Use this to find the Lagrangian of this system. Is this Lagrangian symmetric in x^1? Is momentum conserved in the x^1 direction? Find the equation of motion for the system using the Lagrangian approach. What does the momentum equal? Is this Lagrangian symmetric in x^2? Is momentum conserved in the x^2 direction? Does this make sense physically?

11. For a disk attached to a spring confined to rotate in the plane of the disk about an axis, the potential energy is $V = ½\, k\theta^2$, where θ is the angle of rotation. I is the mass moment of inertia about the axis. What is the Lagrangian of this system? Is this Lagrangian symmetric in θ? Is angular momentum conserved? Find the equation of motion for the system using the Lagrangian approach. What does the angular momentum equal? If there were no spring, would the Lagrangian be symmetric in θ? Would angular momentum be conserved?

12. Show that the Lagrangian density for free Dirac fermions (see Chap. 4) is symmetric under the transformation $\psi \to \psi e^{-i\alpha}$. Use Noether's theorem and the same transformation to show that for Dirac particles, $j^\mu = (\rho, \mathbf{j}) = \bar{\psi}\gamma^\mu\psi$ where $\partial_\mu j^\mu = 0$.

13. Show that for photons $j^\mu = 0$. Do this two ways. i) Assume temporarily that A^μ is complex, so we can write the Lagrangian as $\mathcal{L}_0^{e/m} = -\frac{1}{2}\left(\partial_\nu A_\mu(x)\right)^\dagger \left(\partial^\nu A^\mu(x)\right)$. Use Noether's theorem with the transformation $A^\mu \to A^\mu e^{-i\alpha}$, to obtain j^μ with $\partial_\mu j^\mu = 0$. Then, show that by taking A^μ as real, we must have $j^\mu = 0$. ii) Note that the Lagrangian with real A^μ, $\mathcal{L}_0^{e/m} = -\frac{1}{2}\left(\partial_\nu A_\mu(x)\right)\left(\partial^\nu A^\mu(x)\right)$ is not symmetric under $A^\mu \to A^\mu e^{-i\alpha}$. So, there is no conserved current, i.e., $j^\mu = 0$. In either case, there is always no charge, so $Q = 0$ is conserved.

14. Use Noether's theorem for scalars and the transformation $x^i \to x^i + \alpha^i$ to show that three-momentum k_i is conserved. Then, show the same result via commutation of the three-momentum operator of Chap. 3 with the Hamiltonian.

15. Use Noether's theorem for scalars and the transformation $x^0 \to x^0 + \alpha$ to show that energy $\omega_\mathbf{k}$ is conserved. Is it immediately obvious that you will get the same results from commutation of the energy operator with the Hamiltonian? (Tricky wording here?)

16. Show that the Hamiltonian density for free Dirac fermions is symmetric under the same transformation as in Prob. 12.

17. Is Dirac's field theory a gauge theory? What is the gauge field? Give an example of one of its gauges. What is the gauge transformation? Is this an external or internal symmetry?