Chapter 3 Problem 13 Solution in Detail (and with Sign Error Corrected)

Problem 13. Using (3-100), the expression for 3-momentum in terms of the fields and their conjugate momenta, and the Klein-Gordon field equation solutions, prove (3-101), the number operator form of the 3-momentum operator useful for finding expectation values of 3-momentum. Hint: note that the expectation value <state|a(**k**)a†(**k'**)|state> for **k** not = **k'**, is zero.

Ans. (3-100) is

$$p^i=\int \mathcal{P}^i d^3x=-\int \pi_r\frac{\partial \phi^r}{\partial x^i}d^3x=-\int\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\frac{\partial \phi}{\partial x^i}+\frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger}\frac{\partial \phi^\dagger}{\partial x^i}\right)d^3x \qquad \mathcal{L}=\mathcal{L}_0^0 \text{ here} \tag{3-100}$$

From (3-33), pg. 49,

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}}=\frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}}=\dot{\phi}^\dagger \qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger}=\frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}^\dagger}=\dot{\phi}, \qquad \text{so} \tag{A}$$

$$p^i=-\int\left(\dot{\phi}^\dagger\phi_{,i}+\dot{\phi}\phi^\dagger_{,i}\right)d^3x . \tag{B}$$

From (3-36), pg. 50,

$$\phi(x)=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ikx}+b^\dagger(\mathbf{k})e^{ikx}\right) \qquad \phi^\dagger(x)=\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ikx}+a^\dagger(\mathbf{k})e^{ikx}\right), \text{ so} \tag{3-36}$$

$$\dot{\phi}=\sum_{\mathbf{k}'}\frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(a(\mathbf{k}')e^{-ik'x}-b^\dagger(\mathbf{k}')e^{ik'x}\right) \qquad \dot{\phi}^\dagger=\sum_{\mathbf{k}'}\frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ikx}-a^\dagger(\mathbf{k})e^{ikx}\right)$$

$$\phi_{,i}=\sum_{\mathbf{k}'}\frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(a(\mathbf{k}')e^{-ik'x}-b^\dagger(\mathbf{k}')e^{ik'x}\right) \qquad \phi^\dagger_{,i}=\sum_{\mathbf{k}}\frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ikx}-a^\dagger(\mathbf{k})e^{ikx}\right), \tag{C}$$

where in (C) we have used primed dummy indices in some places to help in what is to come. Using (C) in (B), we have

$$\begin{aligned} p^i=-\int&\left\{\left(\sum_{\mathbf{k}}\frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ikx}-a^\dagger(\mathbf{k})e^{ikx}\right)\right)\left(\sum_{\mathbf{k}'}\frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(a(\mathbf{k}')e^{-ik'x}-b^\dagger(\mathbf{k}')e^{ik'x}\right)\right)\right.\\ &\left.+\left(\sum_{\mathbf{k}'}\frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(a(\mathbf{k}')e^{-ik'x}-b^\dagger(\mathbf{k}')e^{ik'x}\right)\right)\left(\sum_{\mathbf{k}}\frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}}\left(b(\mathbf{k})e^{-ikx}-a^\dagger(\mathbf{k})e^{ikx}\right)\right)\right\}d^3x . \end{aligned} \tag{D}$$

$$\begin{aligned} =-\int&\left\{\sum_{\mathbf{k}}\frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}}\sum_{\mathbf{k}'}\frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(b(\mathbf{k})e^{-ikx}-a^\dagger(\mathbf{k})e^{ikx}\right)\left(a(\mathbf{k}')e^{-ik'x}-b^\dagger(\mathbf{k}')e^{ik'x}\right)\right.\\ &\left.+\sum_{\mathbf{k}'}\frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}\sum_{\mathbf{k}}\frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k}')e^{-ik'x}-b^\dagger(\mathbf{k}')e^{ik'x}\right)\left(b(\mathbf{k})e^{-ikx}-a^\dagger(\mathbf{k})e^{ikx}\right)\right\}d^3x. \end{aligned} \tag{E}$$

We now make the multiplications in (E), and to make it easier to keep track of the various terms, we label them with numbers in boxes above the terms.

$$\begin{aligned} p^i=-\int&\left\{\sum_{\mathbf{k}}\frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}}\sum_{\mathbf{k}'}\frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(\begin{aligned}&\overset{\boxed{1}}{b(\mathbf{k})e^{-ikx}a(\mathbf{k}')e^{-ik'x}}-\overset{\boxed{2}}{b(\mathbf{k})e^{-ikx}b^\dagger(\mathbf{k}')e^{ik'x}}\\ &-\overset{\boxed{3}}{a^\dagger(\mathbf{k})e^{ikx}a(\mathbf{k}')e^{-ik'x}}+\overset{\boxed{4}}{a^\dagger(\mathbf{k})e^{ikx}b^\dagger(\mathbf{k}')e^{ik'x}}\end{aligned}\right)\right.\\ &\left.+\sum_{\mathbf{k}'}\frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}\sum_{\mathbf{k}}\frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}}\left(\begin{aligned}&\overset{\boxed{5}}{a(\mathbf{k}')e^{-ik'x}b(\mathbf{k})e^{-ikx}}-\overset{\boxed{6}}{a(\mathbf{k}')e^{-ik'x}a^\dagger(\mathbf{k})e^{ikx}}\\ &-\overset{\boxed{7}}{b^\dagger(\mathbf{k}')e^{ik'x}b(\mathbf{k})e^{-ikx}}+\overset{\boxed{8}}{b^\dagger(\mathbf{k}')e^{ik'x}a^\dagger(\mathbf{k})e^{ikx}}\end{aligned}\right)\right\}d^3x. \end{aligned} \tag{F}$$

Now, so that we can combine the rows in (F) in an advantageous way, exchange the **k** with **k′** in the last two rows of (F),

$$p^i = -\int \left\{ \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \begin{pmatrix} \overset{\boxed{1}}{b(\mathbf{k})e^{-ikx}a(\mathbf{k}')e^{-ik'x}} \overset{\boxed{2}}{-b(\mathbf{k})e^{-ikx}b^\dagger(\mathbf{k}')e^{ik'x}} \\ \overset{\boxed{3}}{-a^\dagger(\mathbf{k})e^{ikx}a(\mathbf{k}')e^{-ik'x}} \overset{\boxed{4}}{+a^\dagger(\mathbf{k})e^{ikx}b^\dagger(\mathbf{k}')e^{ik'x}} \end{pmatrix} + \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \begin{pmatrix} \overset{\boxed{5}}{a(\mathbf{k})e^{-ikx}b(\mathbf{k}')e^{-ik'x}} \overset{\boxed{6}}{-a(\mathbf{k})e^{-ikx}a^\dagger(\mathbf{k}')e^{ik'x}} \\ \overset{\boxed{7}}{-b^\dagger(\mathbf{k})e^{ikx}b(\mathbf{k}')e^{-ik'x}} \overset{\boxed{8}}{+b^\dagger(\mathbf{k})e^{ikx}a^\dagger(\mathbf{k}')e^{ik'x}} \end{pmatrix} \right\} d^3x. \quad \text{(G)}$$

$$p^i = -\int \left\{ \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \begin{pmatrix} \overset{\boxed{1}}{b(\mathbf{k})e^{-ikx}a(\mathbf{k}')e^{-ik'x}} \overset{\boxed{5}}{+ a(\mathbf{k})e^{-ikx}b(\mathbf{k}')e^{-ik'x}} \\ \overset{\boxed{4}}{+a^\dagger(\mathbf{k})e^{ikx}b^\dagger(\mathbf{k}')e^{ik'x}} \overset{\boxed{8}}{+b^\dagger(\mathbf{k})e^{ikx}a^\dagger(\mathbf{k}')e^{ik'x}} \end{pmatrix} + \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \begin{pmatrix} \overset{\boxed{2}}{-b(\mathbf{k})e^{-ikx}b^\dagger(\mathbf{k}')e^{ik'x}} \overset{\boxed{6}}{-a(\mathbf{k})e^{-ikx}a^\dagger(\mathbf{k}')e^{ik'x}} \\ \overset{\boxed{3}}{-a^\dagger(\mathbf{k})e^{ikx}a(\mathbf{k}')e^{-ik'x}} \overset{\boxed{7}}{-b^\dagger(\mathbf{k})e^{ikx}b(\mathbf{k}')e^{-ik'x}} \end{pmatrix} \right\} d^3x. \quad \text{(H)}$$

In the first two rows of (H) the integration over **x** will yield zero unless **k′** = –**k**. In the third and fourth rows, the integration will yield zero unless **k′** = **k**. Thus, where we move the k^i in the first two rows inside the parentheses for a reason to become apparent.

$$p^i = -V \left\{ \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \frac{-i}{\sqrt{2V\omega_{\mathbf{k}}}} \begin{pmatrix} \overset{\boxed{1}}{k^i b(\mathbf{k})a(-\mathbf{k})e^{-i2\omega_{\mathbf{k}}t}} \overset{\boxed{5}}{+ k^i a(\mathbf{k})b(-\mathbf{k})e^{-i2\omega_{\mathbf{k}}t}} \\ \overset{\boxed{4}}{+k^i a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})e^{i2\omega_{\mathbf{k}}t}} \overset{\boxed{8}}{+k^i b^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})e^{i2\omega_{\mathbf{k}}t}} \end{pmatrix} + \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}} \begin{pmatrix} \overset{\boxed{2}}{-b(\mathbf{k})b^\dagger(\mathbf{k})} \quad \overset{\boxed{6}}{-a(\mathbf{k})a^\dagger(\mathbf{k})} \\ \overset{\boxed{3}}{-a^\dagger(\mathbf{k})a(\mathbf{k})} \quad \overset{\boxed{7}}{-b^\dagger(\mathbf{k})b(\mathbf{k})} \end{pmatrix} \right\}. \quad \text{(I)}$$

$$p^i = \sum_{\mathbf{k}} \frac{1}{2} \begin{pmatrix} \overset{\boxed{1}}{k^i b(\mathbf{k})a(-\mathbf{k})e^{-i2\omega_{\mathbf{k}}t}} \overset{\boxed{5}}{+ k^i a(\mathbf{k})b(-\mathbf{k})e^{-i2\omega_{\mathbf{k}}t}} \\ \overset{\boxed{4}}{+k^i a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k})e^{i2\omega_{\mathbf{k}}t}} \overset{\boxed{8}}{+k^i b^\dagger(\mathbf{k})a^\dagger(-\mathbf{k})e^{i2\omega_{\mathbf{k}}t}} \end{pmatrix} - \sum_{\mathbf{k}} \frac{k^i}{2} \left(\overset{\boxed{2}}{-b(\mathbf{k})b^\dagger(\mathbf{k})} \overset{\boxed{6}}{-a(\mathbf{k})a^\dagger(\mathbf{k})} \overset{\boxed{3}}{-a^\dagger(\mathbf{k})a(\mathbf{k})} \overset{\boxed{7}}{-b^\dagger(\mathbf{k})b(\mathbf{k})} \right). \quad \text{(J)}$$

Note that in the sum of the first two rows of (J), we get one – k^i for every k^i. Thus, for the term in the sum labeled $\boxed{1}$ for k^i and the term labeled $\boxed{5}$ for – k^i, we have, due to the commutation relations,

$$\overset{\boxed{1}}{k^i b(\mathbf{k})a(-\mathbf{k})e^{-i2\omega_{\mathbf{k}}t}} - \overset{\boxed{5}}{k^i a(-\mathbf{k})b(\mathbf{k})e^{-i2\omega_{\mathbf{k}}t}} = k^i \underbrace{\left(b(\mathbf{k})a(-\mathbf{k}) - a(-\mathbf{k})b(\mathbf{k})\right)}_{-\left[a(-\mathbf{k}),b(\mathbf{k})\right]=0} e^{-i2\omega_{\mathbf{k}}t} = 0. \quad \text{(K)}$$

In similar fashion all terms in the first two rows of (J) vanish, leaving

$$p^i = \sum_{\mathbf{k}} \frac{k^i}{2}\left(\underbrace{\overset{\boxed{2}}{b(\mathbf{k})b^\dagger(\mathbf{k})}}_{b^\dagger(\mathbf{k})b(\mathbf{k})+1} + \underbrace{\overset{\boxed{6}}{a(\mathbf{k})a^\dagger(\mathbf{k})}}_{a^\dagger(\mathbf{k})a(\mathbf{k})+1} + \overset{\boxed{3}}{a^\dagger(\mathbf{k})a(\mathbf{k})} + \overset{\boxed{7}}{b^\dagger(\mathbf{k})b(\mathbf{k})} \right). \tag{L}$$

$$= \sum_{\mathbf{k}} k^i \left(N_a(\mathbf{k}) + \frac{1}{2} + N_b(\mathbf{k}) + \frac{1}{2} \right).$$

For every k^i in the summation there is an opposite direction 3-momentum of $-k^i$, so the ½ terms cancel out. Converting from index notation to bold 3-vector notation, we have (3-101).

$$\mathbf{P} = \int_V \not{\mathbf{p}}\, d^3x = \sum_{\mathbf{k}} \mathbf{k}\left(N_a(\mathbf{k}) + N_b(\mathbf{k}) \right). \tag{3-101}$$