

Chapter 3 Problem 13 Solution in Detail (and with Sign Error Corrected)

Problem 13. Using (3-100), the expression for 3-momentum in terms of the fields and their conjugate momenta, and the Klein-Gordon field equation solutions, prove (3-101), the number operator form of the 3-momentum operator useful for finding expectation values of 3-momentum. Hint: note that the expectation value $\langle \text{state} | a(\mathbf{k}) a^\dagger(\mathbf{k}') | \text{state} \rangle$ for $\mathbf{k} \text{ not } = \mathbf{k}'$, is zero.

Ans. (3-100) is

$$p^i = \int \not{\mathcal{L}}^i d^3x = - \int \pi_r \frac{\partial \phi^r}{\partial x^i} d^3x = - \int \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}} \frac{\partial \phi}{\partial x^i} + \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} \frac{\partial \phi^\dagger}{\partial x^i} \right) d^3x \quad \mathcal{L} = \mathcal{L}_0^0 \text{ here} \quad (3-100)$$

From (3-33), pg. 49,

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}} = \dot{\phi}^\dagger \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}^\dagger} = \frac{\partial \mathcal{L}_0^0}{\partial \dot{\phi}^\dagger} = \dot{\phi}, \quad \text{so} \quad (A)$$

$$p^i = - \int (\dot{\phi}^\dagger \phi_{,i} + \dot{\phi} \phi_{,i}^\dagger) d^3x. \quad (B)$$

From (3-36), pg. 50,

$$\phi(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} (a(\mathbf{k}) e^{-ikx} + b^\dagger(\mathbf{k}) e^{ikx}) \quad \phi^\dagger(x) = \sum_{\mathbf{k}} \frac{1}{\sqrt{2V\omega_{\mathbf{k}}}} (b(\mathbf{k}) e^{-ikx} + a^\dagger(\mathbf{k}) e^{ikx}), \text{ so} \quad (3-36)$$

$$\dot{\phi} = \sum_{\mathbf{k}'} \frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}} (a(\mathbf{k}') e^{-ik'x} - b^\dagger(\mathbf{k}') e^{ik'x}) \quad \dot{\phi}^\dagger = \sum_{\mathbf{k}'} \frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}} (b(\mathbf{k}') e^{-ik'x} - a^\dagger(\mathbf{k}') e^{ik'x}) \quad (C)$$

$$\phi_{,i} = \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} (a(\mathbf{k}') e^{-ik'x} - b^\dagger(\mathbf{k}') e^{ik'x}) \quad \phi_{,i}^\dagger = \sum_{\mathbf{k}} \frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}} (b(\mathbf{k}) e^{-ikx} - a^\dagger(\mathbf{k}) e^{ikx}),$$

where in (C) we have used primed dummy indices in some places to help in what is to come. Using (C) in (B), we have

$$p^i = - \int \left\{ \left(\sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} (b(\mathbf{k}) e^{-ikx} - a^\dagger(\mathbf{k}) e^{ikx}) \right) \left(\sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} (a(\mathbf{k}') e^{-ik'x} - b^\dagger(\mathbf{k}') e^{ik'x}) \right) \right. \\ \left. + \left(\sum_{\mathbf{k}'} \frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}} (a(\mathbf{k}') e^{-ik'x} - b^\dagger(\mathbf{k}') e^{ik'x}) \right) \left(\sum_{\mathbf{k}} \frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}} (b(\mathbf{k}) e^{-ikx} - a^\dagger(\mathbf{k}) e^{ikx}) \right) \right\} d^3x. \quad (D)$$

$$= - \int \left\{ \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} (b(\mathbf{k}) e^{-ikx} - a^\dagger(\mathbf{k}) e^{ikx}) (a(\mathbf{k}') e^{-ik'x} - b^\dagger(\mathbf{k}') e^{ik'x}) \right. \\ \left. + \sum_{\mathbf{k}'} \frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}} \sum_{\mathbf{k}} \frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}} (a(\mathbf{k}') e^{-ik'x} - b^\dagger(\mathbf{k}') e^{ik'x}) (b(\mathbf{k}) e^{-ikx} - a^\dagger(\mathbf{k}) e^{ikx}) \right\} d^3x. \quad (E)$$

We now make the multiplications in (E), and to make it easier to keep track of the various terms, we label them with numbers in boxes above the terms.

$$p^i = - \int \left\{ \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \left(\begin{array}{l} b(\mathbf{k}) e^{-ikx} \overset{\boxed{1}}{a(\mathbf{k}') e^{-ik'x}} - b(\mathbf{k}) e^{-ikx} \overset{\boxed{2}}{b^\dagger(\mathbf{k}') e^{ik'x}} \\ - a^\dagger(\mathbf{k}) e^{ikx} \overset{\boxed{3}}{a(\mathbf{k}') e^{-ik'x}} + a^\dagger(\mathbf{k}) e^{ikx} \overset{\boxed{4}}{b^\dagger(\mathbf{k}') e^{ik'x}} \end{array} \right) \right. \\ \left. + \sum_{\mathbf{k}'} \frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}} \sum_{\mathbf{k}} \frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}} \left(\begin{array}{l} a(\mathbf{k}') e^{-ik'x} \overset{\boxed{5}}{b(\mathbf{k}) e^{-ikx}} - a(\mathbf{k}') e^{-ik'x} \overset{\boxed{6}}{a^\dagger(\mathbf{k}) e^{ikx}} \\ - b^\dagger(\mathbf{k}') e^{ik'x} \overset{\boxed{7}}{b(\mathbf{k}) e^{-ikx}} + b^\dagger(\mathbf{k}') e^{ik'x} \overset{\boxed{8}}{a^\dagger(\mathbf{k}) e^{ikx}} \end{array} \right) \right\} d^3x. \quad (F)$$

Now, so that we can combine the rows in (F) in an advantageous way, exchange the \mathbf{k} with \mathbf{k}' in the last two rows of (F),

$$p^i = -\int \left\{ \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \left(\begin{array}{l} b(\mathbf{k})e^{-ikx} \overset{\boxed{1}}{a(\mathbf{k}')e^{-ik'x}} - b(\mathbf{k})e^{-ikx} \overset{\boxed{2}}{b^\dagger(\mathbf{k}')e^{ik'x}} \\ - a^\dagger(\mathbf{k})e^{ikx} \overset{\boxed{3}}{a(\mathbf{k}')e^{-ik'x}} + a^\dagger(\mathbf{k})e^{ikx} \overset{\boxed{4}}{b^\dagger(\mathbf{k}')e^{ik'x}} \end{array} \right) \right. \\ \left. + \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \left(\begin{array}{l} a(\mathbf{k})e^{-ikx} \overset{\boxed{5}}{b(\mathbf{k}')e^{-ik'x}} - a(\mathbf{k})e^{-ikx} \overset{\boxed{6}}{a^\dagger(\mathbf{k}')e^{ik'x}} \\ - b^\dagger(\mathbf{k})e^{ikx} \overset{\boxed{7}}{b(\mathbf{k}')e^{-ik'x}} + b^\dagger(\mathbf{k})e^{ikx} \overset{\boxed{8}}{a^\dagger(\mathbf{k}')e^{ik'x}} \end{array} \right) \right\} d^3x. \quad (\text{G})$$

$$p^i = -\int \left\{ \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \left(\begin{array}{l} b(\mathbf{k})e^{-ikx} \overset{\boxed{1}}{a(\mathbf{k}')e^{-ik'x}} + a(\mathbf{k})e^{-ikx} \overset{\boxed{5}}{b(\mathbf{k}')e^{-ik'x}} \\ + a^\dagger(\mathbf{k})e^{ikx} \overset{\boxed{4}}{b^\dagger(\mathbf{k}')e^{ik'x}} + b^\dagger(\mathbf{k})e^{ikx} \overset{\boxed{8}}{a^\dagger(\mathbf{k}')e^{ik'x}} \end{array} \right) \right. \\ \left. + \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \sum_{\mathbf{k}'} \frac{ik'^i}{\sqrt{2V\omega_{\mathbf{k}'}}} \left(\begin{array}{l} -b(\mathbf{k})e^{-ikx} \overset{\boxed{2}}{b^\dagger(\mathbf{k}')e^{ik'x}} - a(\mathbf{k})e^{-ikx} \overset{\boxed{6}}{a^\dagger(\mathbf{k}')e^{ik'x}} \\ - a^\dagger(\mathbf{k})e^{ikx} \overset{\boxed{3}}{a(\mathbf{k}')e^{-ik'x}} - b^\dagger(\mathbf{k})e^{ikx} \overset{\boxed{7}}{b(\mathbf{k}')e^{-ik'x}} \end{array} \right) \right\} d^3x. \quad (\text{H})$$

In the first two rows of (H) the integration over \mathbf{x} will yield zero unless $\mathbf{k}' = -\mathbf{k}$. In the third and fourth rows, the integration will yield zero unless $\mathbf{k}' = \mathbf{k}$. Thus, where we move the k^i in the first two rows inside the parentheses for a reason to become apparent.

$$p^i = -V \int \left\{ \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \frac{-i}{\sqrt{2V\omega_{\mathbf{k}}}} \left(\begin{array}{l} k^i b(\mathbf{k})a(-\mathbf{k}) \overset{\boxed{1}}{e^{-i2\omega_{\mathbf{k}}t}} + k^i a(\mathbf{k})b(-\mathbf{k}) \overset{\boxed{5}}{e^{-i2\omega_{\mathbf{k}}t}} \\ + k^i a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k}) \overset{\boxed{4}}{e^{i2\omega_{\mathbf{k}}t}} + k^i b^\dagger(\mathbf{k})a^\dagger(-\mathbf{k}) \overset{\boxed{8}}{e^{i2\omega_{\mathbf{k}}t}} \end{array} \right) \right. \\ \left. + \sum_{\mathbf{k}} \frac{-i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}} \frac{ik^i}{\sqrt{2V\omega_{\mathbf{k}}}} \left(\begin{array}{l} -b(\mathbf{k})b^\dagger(\mathbf{k}) - a(\mathbf{k})a^\dagger(\mathbf{k}) \\ - a^\dagger(\mathbf{k})a(\mathbf{k}) - b^\dagger(\mathbf{k})b(\mathbf{k}) \end{array} \right) \right\} d^3x. \quad (\text{I})$$

$$p^i = \sum_{\mathbf{k}} \frac{1}{2} \left(\begin{array}{l} k^i b(\mathbf{k})a(-\mathbf{k}) \overset{\boxed{1}}{e^{-i2\omega_{\mathbf{k}}t}} + k^i a(\mathbf{k})b(-\mathbf{k}) \overset{\boxed{5}}{e^{-i2\omega_{\mathbf{k}}t}} \\ + k^i a^\dagger(\mathbf{k})b^\dagger(-\mathbf{k}) \overset{\boxed{4}}{e^{i2\omega_{\mathbf{k}}t}} + k^i b^\dagger(\mathbf{k})a^\dagger(-\mathbf{k}) \overset{\boxed{8}}{e^{i2\omega_{\mathbf{k}}t}} \end{array} \right) \\ - \sum_{\mathbf{k}} \frac{k^i}{2} \left(\begin{array}{l} -b(\mathbf{k})b^\dagger(\mathbf{k}) - a(\mathbf{k})a^\dagger(\mathbf{k}) \\ - a^\dagger(\mathbf{k})a(\mathbf{k}) - b^\dagger(\mathbf{k})b(\mathbf{k}) \end{array} \right). \quad (\text{J})$$

Note that in the sum of the first two rows of (J), we get one $-k^i$ for every k^i . Thus, for the term in the sum labeled $\boxed{1}$ for k^i and the term labeled $\boxed{5}$ for $-k^i$, we have, due to the commutation relations,

$$k^i b(\mathbf{k})a(-\mathbf{k}) \overset{\boxed{1}}{e^{-i2\omega_{\mathbf{k}}t}} - k^i a(-\mathbf{k})b(\mathbf{k}) \overset{\boxed{5}}{e^{-i2\omega_{\mathbf{k}}t}} = k^i \underbrace{(b(\mathbf{k})a(-\mathbf{k}) - a(-\mathbf{k})b(\mathbf{k}))}_{-[a(-\mathbf{k})b(\mathbf{k})]=0} e^{-i2\omega_{\mathbf{k}}t} = 0. \quad (\text{K})$$

In similar fashion all terms in the first two rows of (J) vanish, leaving

Chapter 3 Problem Solutions

$$\begin{aligned}
 p^i &= \sum_{\mathbf{k}} \frac{k^i}{2} \left(\underbrace{b(\mathbf{k})b^\dagger(\mathbf{k})}_{b^\dagger(\mathbf{k})b(\mathbf{k})+1} + \underbrace{a(\mathbf{k})a^\dagger(\mathbf{k})}_{a^\dagger(\mathbf{k})a(\mathbf{k})+1} + a^\dagger(\mathbf{k})a(\mathbf{k}) + b^\dagger(\mathbf{k})b(\mathbf{k}) \right) \\
 &= \sum_{\mathbf{k}} k^i \left(N_a(\mathbf{k}) + \frac{1}{2} + N_b(\mathbf{k}) + \frac{1}{2} \right).
 \end{aligned} \tag{L}$$

For every k^i in the summation there is an opposite direction 3-momentum of $-k^i$, so the $\frac{1}{2}$ terms cancel out. Converting from index notation to bold 3-vector notation, we have (3-101).

$$\mathbf{P} = \int_V \boldsymbol{\rho} d^3x = \sum_{\mathbf{k}} \mathbf{k} (N_a(\mathbf{k}) + N_b(\mathbf{k})). \tag{3-101}$$