<u>Proof of coefficient commutation relations</u>

To prove (3-41), start with (3-40) and take different spatial coordinates **x** and **y**, but the same time coordinate *t*, for $\phi$ and $\pi^0{}_0$. This results in the <u>equal time commutation relations</u>

$$\left[\phi(\mathbf{x},t)\pi_0^0(\mathbf{y},t)-\pi_0^0(\mathbf{y},t)\phi(\mathbf{x},t)\right]=\left[\phi(\mathbf{x},t)\dot{\phi}^{\dagger}(\mathbf{y},t)-\dot{\phi}^{\dagger}(\mathbf{y},t)\phi(\mathbf{x},t)\right]=i\delta(\mathbf{x}-\mathbf{y}), \qquad (3\text{-}42)$$

*Proving coefficient commutation relations*

which are only important at this point as a step in our proof. Then, plugging the discrete solutions (3-36) into the middle part of (3-42), where to save space we use the compressed notation $a_{\mathbf{k}} = a(\mathbf{k})$, etc., we get

$$\left(\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}b^{\dagger}_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}\right)\left(\sum_{\mathbf{k}'}\frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}b_{\mathbf{k}'}e^{-i(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{y})}+\sum_{\mathbf{k}'}\frac{i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}a^{\dagger}_{\mathbf{k}'}e^{i(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{y})}\right)$$
$$-\left(\sum_{\mathbf{k}'}\frac{-i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}b_{\mathbf{k}'}e^{-i(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{y})}+\sum_{\mathbf{k}'}\frac{i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}a^{\dagger}_{\mathbf{k}'}e^{i(\omega_{\mathbf{k}'}t-\mathbf{k}'\cdot\mathbf{y})}\right)\left(\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}+\sum_{\mathbf{k}}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}b^{\dagger}_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}t-\mathbf{k}\cdot\mathbf{x})}\right)$$
$$=\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\frac{i\omega_{\mathbf{k}'}}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\begin{pmatrix}-a_{\mathbf{k}}b_{\mathbf{k}'}e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}+a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}\\-b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}'}e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}+b^{\dagger}_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}\\+b_{\mathbf{k}'}a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}+b_{\mathbf{k}'}b^{\dagger}_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}\\-a^{\dagger}_{\mathbf{k}'}a_{\mathbf{k}}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}-a^{\dagger}_{\mathbf{k}'}b^{\dagger}_{\mathbf{k}}e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}\end{pmatrix}=i\delta(\mathbf{x}-\mathbf{y}). \qquad (3\text{-}43)$$

*K-G solutions into equal time commutator*

Using the math identity for the 3D Dirac delta function

$$\delta(\mathbf{x}-\mathbf{y})=\frac{1}{V}\sum_{n=-\infty}^{+\infty}e^{-i\mathbf{k}_n\cdot(\mathbf{x}-\mathbf{y})}\qquad\begin{pmatrix}\text{in our notation}=\dfrac{1}{V}\displaystyle\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\\ \text{or equivalently, }\dfrac{1}{2V}\displaystyle\sum_{\mathbf{k}}\left(e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\right)\end{pmatrix} \qquad (3\text{-}44)$$

*Re-express Dirac delta function*

on the RHS of the last row in (3-43), and matching terms, we see that all terms where $\mathbf{k}' \neq \pm\,\mathbf{k}$ must equal zero, since (3-44) has no terms in both **k** and **k′**. These particular terms reduce to the following form, summed over **k** and **k′**.

$$\frac{i\omega_{\mathbf{k}'}}{2V\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}}\begin{pmatrix}\underbrace{\left(b_{\mathbf{k}'}a_{\mathbf{k}}-a_{\mathbf{k}}b_{\mathbf{k}'}\right)}_{\text{must}=0}e^{-i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}+\underbrace{\left(a_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}-a^{\dagger}_{\mathbf{k}'}a_{\mathbf{k}}\right)}_{\text{must}=0}e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}\\+\underbrace{\left(b_{\mathbf{k}'}b^{\dagger}_{\mathbf{k}}-b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}'}\right)}_{\text{must}=0}e^{i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}-\mathbf{k}'\cdot\mathbf{y})}+\underbrace{\left(b^{\dagger}_{\mathbf{k}}a^{\dagger}_{\mathbf{k}'}-a^{\dagger}_{\mathbf{k}'}b^{\dagger}_{\mathbf{k}}\right)}_{\text{must}=0}e^{i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})t}e^{-i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{y})}\end{pmatrix}=0 \qquad (3\text{-}45)$$

$$\left(\text{All terms in summations with }\mathbf{k}'\neq\pm\mathbf{k}\text{ equal 0, as no terms on RHS in }\mathbf{k}\text{ and }\mathbf{k}'\right)$$

*Terms where* $\mathbf{k}' \neq \pm\,\mathbf{k}$

*These must vanish, so their commutators must = 0*

So, all possible coefficient commutators with $\mathbf{k}' \neq \mathbf{k}$ or $-\,\mathbf{k}$ vanish. The remaining terms all have $\mathbf{k}' = \pm\,\mathbf{k}$, which means $\omega_{\mathbf{k}} = \omega_{\mathbf{k}'}$. Some of these have an exponential form $i(\omega_{\mathbf{k}} + \omega_{\mathbf{k}'})t$, and those terms give us a summation of terms over **k** having form, for each possible **k′**, of

$$\frac{i\omega_{\mathbf{k}}}{2V\omega_{\mathbf{k}}}\begin{pmatrix}\underbrace{\left(b_{\mathbf{k}}a_{\mathbf{k}}-a_{\mathbf{k}}b_{\mathbf{k}}\right)}_{\text{must}=0}\underbrace{e^{-i2\omega_{\mathbf{k}}t}}_{\neq 0}e^{i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}+\underbrace{\left(b^{\dagger}_{\mathbf{k}}a^{\dagger}_{\mathbf{k}}-a^{\dagger}_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}\right)}_{\text{must}=0}e^{i2\omega_{\mathbf{k}}t}e^{-i\mathbf{k}\cdot(\mathbf{x}+\mathbf{y})}\qquad(\leftarrow\mathbf{k}'=\mathbf{k})\\+\underbrace{\left(b_{-\mathbf{k}}a_{\mathbf{k}}-a_{\mathbf{k}}b_{-\mathbf{k}}\right)}_{\text{must}=0}e^{-i2\omega_{\mathbf{k}}t}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}+\underbrace{\left(b^{\dagger}_{\mathbf{k}}a^{\dagger}_{-\mathbf{k}}-a^{\dagger}_{-\mathbf{k}}b^{\dagger}_{\mathbf{k}}\right)}_{\text{must}=0}e^{i2\omega_{\mathbf{k}}t}e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\qquad(\leftarrow\mathbf{k}'=-\mathbf{k})\end{pmatrix}=0 \qquad (3\text{-}46)$$

*Terms where* $\mathbf{k}' = \pm\,\mathbf{k}$*, i.e., those of form* $exp(i(\omega_{\mathbf{k}}+\omega_{\mathbf{k}})t)$

$$\left(\text{All time dependent terms with }\mathbf{k}'=\pm\mathbf{k}\text{ equal 0, as no time dependence on RHS}\right)$$

*Commutators must = 0*

For these terms, the coefficient commutators must vanish because the exponential in $\omega_{\mathbf{k}}$ varies in time, whereas there is no such variation on the RHS of the last row in (3-43).

The remaining terms have exponential form $i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})t$ and $\mathbf{k}' = \pm\,\mathbf{k}$. Adding those terms for $\mathbf{k}' = \mathbf{k}$ with the terms for $\mathbf{k}' = -\mathbf{k}$ yields, with the relevant terms on the RHS of (3-43) (see 2nd row in parentheses of (3-44)) on the RHS below,

*Remaining terms where* $\mathbf{k}' = \pm\,\mathbf{k}$*, i.e., those of form* $exp(i(\omega_{\mathbf{k}} - \omega_{\mathbf{k}})t)$

$$\frac{i}{2V}\begin{pmatrix} \underbrace{\left(a_{\mathbf{k}}a_{\mathbf{k}}^{\dagger}-a_{\mathbf{k}}^{\dagger}a_{\mathbf{k}}\right)}_{\text{must}=1}\underbrace{e^{-i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}})t}}_{=1}e^{i\mathbf{k}\bullet(\mathbf{x}-\mathbf{y})}+\underbrace{\left(b_{\mathbf{k}}b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}\right)}_{\text{must}=1}e^{-i\mathbf{k}\bullet(\mathbf{x}-\mathbf{y})} \quad (\leftarrow \mathbf{k}'=\mathbf{k}) \\ +\underbrace{\left(a_{\mathbf{k}}a_{-\mathbf{k}}^{\dagger}-a_{-\mathbf{k}}^{\dagger}a_{\mathbf{k}}\right)}_{\text{must}=0}e^{i\mathbf{k}\bullet(\mathbf{x}+\mathbf{y})}+\underbrace{\left(b_{-\mathbf{k}}b_{\mathbf{k}}^{\dagger}-b_{\mathbf{k}}^{\dagger}b_{-\mathbf{k}}\right)}_{\text{must}=0}e^{-i\mathbf{k}\bullet(\mathbf{x}+\mathbf{y})} \quad (\leftarrow\mathbf{k}'=-\mathbf{k})\end{pmatrix} = \frac{i}{2V}\begin{pmatrix}e^{i\mathbf{k}\bullet(\mathbf{x}-\mathbf{y})}\\ +e^{-i\mathbf{k}\bullet(\mathbf{x}-\mathbf{y})}\end{pmatrix} \quad \text{(3-47)}$$

*Key commutators must* = 1

$\left(\text{All time independent terms in summation with } \mathbf{k}' = \pm\,\mathbf{k} \text{ must equal RHS}\right).$

All terms with ($\mathbf{x} + \mathbf{y}$) in the exponents of the LHS must equal zero, as the RHS only has terms in ($\mathbf{x} - \mathbf{y}$). The only way the LHS of (3-47) matches the RHS is if each coefficient commutator in the first row equals unity.

The commutation relations for $a_{\mathbf{k}}a_{\mathbf{k}'}^{\dagger}$ and $b_{\mathbf{k}}b_{\mathbf{k}'}^{\dagger}$ in (3-45) to (3-47) are the same as (3-41). QED.

If you are ambitious, have extra time, and/or simply have to prove everything to yourself, do Prob. 7 to derive the continuous solution commutators of (3-41).

<u>End of coefficient commutation relations proof</u>

With the coefficient commutator relations in hand, we are finally ready to dive into the real core of QFT.

### *3.4 The Hamiltonian in QFT*

We find the Hamiltonian by integrating the Hamiltonian density $\mathcal{H}$ over all space (a volume $V$ containing the discrete solutions, which we can make as large as we like.) In QFT, we express $\mathcal{H}$ in terms of a complex field and substitute our field equation solutions.

$H = \int \mathcal{H}\, dV$

#### *3.4.1 The Free Scalar Hamiltonian in Terms of the Coefficients*

For a free scalar field $\mathcal{H} = \mathcal{H}_0^{\;0}$, as in (3-33), where we employ our discrete, plane wave solutions (3-36) we get

*$H = \int \mathcal{H}\, dV$ in terms of the fields*

$$H_0^0 = \int \mathcal{H}_0^0 d^3x = \int\left(\dot{\phi}\dot{\phi}^{\dagger} + \nabla\phi^{\dagger}\cdot\nabla\phi + \mu^2\phi^{\dagger}\phi\right)d^3x =$$

$$\int\left(\sum_{\mathbf{k}}\frac{\partial}{\partial t}\frac{1}{\sqrt{2V\omega_{\mathbf{k}}}}\left(a(\mathbf{k})e^{-ikx}+b^{\dagger}(\mathbf{k})e^{ikx}\right)\right)\left(\sum_{\mathbf{k}'}\frac{\partial}{\partial t}\frac{1}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(b(\mathbf{k}')e^{-ik'x}+a^{\dagger}(\mathbf{k}')e^{ik'x}\right)\right)d^3x \quad \text{(3-48)}$$

$$+\int\left(-\partial_i\phi^{\dagger}\partial^i\phi + \mu^2\phi^{\dagger}\phi\right)d^3x.$$

*Deriving H in terms of the coefficients* ↓

The middle line of (3-48), i.e., the $\int \dot{\phi}\dot{\phi}^{\dagger} d^3x$ part, becomes

$$\int\left(\sum_{\mathbf{k}}\frac{i\omega_{\mathbf{k}}}{\sqrt{2V\omega_{\mathbf{k}}}}\left(-a(\mathbf{k})e^{-ikx}+b^{\dagger}(\mathbf{k})e^{ikx}\right)\right)\left(\sum_{\mathbf{k}'}\frac{i\omega_{\mathbf{k}'}}{\sqrt{2V\omega_{\mathbf{k}'}}}\left(-b(\mathbf{k}')e^{-ik'x}+a^{\dagger}(\mathbf{k}')e^{ik'x}\right)\right)d^3x. \quad \text{(3-49)}$$

or

$$\sum_{\mathbf{k}}\sum_{\mathbf{k}'}\frac{-\sqrt{\omega_{\mathbf{k}}}\sqrt{\omega_{\mathbf{k}'}}}{2V}\int\begin{pmatrix}a(\mathbf{k})b(\mathbf{k}')e^{-ikx}e^{-ik'x}-a(\mathbf{k})a^{\dagger}(\mathbf{k}')e^{-ikx}e^{ik'x}\\ -b^{\dagger}(\mathbf{k})b(\mathbf{k}')e^{ikx}e^{-ik'x}+b^{\dagger}(\mathbf{k})a^{\dagger}(\mathbf{k}')e^{ikx}e^{ik'x}\end{pmatrix}d^3x\,. \quad \text{(3-50)}$$

The sum over $\mathbf{k}$ and $\mathbf{k}'$ is from negative infinity to positive infinity in the x, y, and z directions.

All terms in the integration in (3-50) result in zero except when $\mathbf{k}' = \mathbf{k}$ or $\mathbf{k}' = -\,\mathbf{k}$, because we are integrating orthogonal functions between their boundaries. (This is similar to $sin(2X)sin(4X)$ integrated with respect to $X$ along a complete number of wavelengths, where here $\mathbf{k} = 2$ and $\mathbf{k}' = 4$.) Since the volume of integration in (3-50) equals $V$, we end up with

$$\begin{aligned}\int\dot{\phi}\dot{\phi}^{\dagger}d^3x &= \sum_{\mathbf{k}}\frac{\omega_{\mathbf{k}}}{2}\left(-a(\mathbf{k})b(-\mathbf{k})e^{-2i\omega t}+a(\mathbf{k})a^{\dagger}(\mathbf{k})+b^{\dagger}(\mathbf{k})b(\mathbf{k})-b^{\dagger}(\mathbf{k})a^{\dagger}(-\mathbf{k})e^{2i\omega t}\right)\\ &= \sum_{\mathbf{k}}\frac{(\omega_{\mathbf{k}})^2}{2\omega_{\mathbf{k}}}\left(-a(-\mathbf{k})b(\mathbf{k})e^{-2i\omega t}+a(\mathbf{k})a^{\dagger}(\mathbf{k})+b^{\dagger}(\mathbf{k})b(\mathbf{k})-b^{\dagger}(-\mathbf{k})a^{\dagger}(\mathbf{k})e^{2i\omega t}\right).\end{aligned} \quad \text{(3-51)}$$