Box 2-2 Subtleties

We first need to recall the definition of physical 3-momentum

$$p^i = m\dot{x}^i = m\frac{dx^i}{dt} \qquad \begin{cases}\text{contraviant form is physical 3-momentum we measure} \\ \quad \text{(found from } x^i \text{, the physical coords we measure)}\end{cases}$$

$$p_i = m\dot{x}_i = m\frac{dx_i}{dt} = -m\frac{dx^i}{dt} = -m\dot{x}^i \qquad \begin{cases}\text{covariant form is negative of physical 3-momentum} \\ \quad \text{(is not what we measure for 3-momentum)}\end{cases} \tag{1}$$

Then consider our free total Lagrangian L (no potential energy)

$$L = \frac{1}{2}m\left(\dot{x}^i\right)^2, \tag{2}$$

and its conjugate momentum (as usually defined in tensor analysis so the subscript i before the first equal sign is equivalent to a superscript i in the denominator after the equal sign),

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m\dot{x}^i \qquad \{\text{conjugate momentum via tensor analysis}. \tag{3}$$

Note that the p_i in (3) differs from that in the second row of (1)! In reality, the value of the conjugate momentum equals our physical 3-momentum (top row of (1).

So what we show in the LHS of (B2-2.1) in Box 2-2 is the conjugate momentum p_i, but its actual value $m\dot{x}^i$ is what we use the symbol p^i for in the rest of the book, i.e., the real physical 3-momentum (not its negative).

So to stay consistent with the symbolism of the rest of the book (and in the literature) we really should have written (B-2-2.1) as

$$p_i = -\frac{\partial L}{\partial \dot{x}^i} \xrightarrow[\text{divide by particle volume}]{\text{for small particle in medium,}} \not{p}_i = -\frac{\partial \mathcal{L}}{\partial \dot{x}^i} \qquad \begin{cases}\text{(B2-2.1) re-written in terms of} \\ \text{symbols used elsewhere in book,} \\ \text{opposite sign of tensor analysis}\end{cases} \tag{4}$$

(This would have been inconsistent with the usual tensor analysis notation.)

Similar logic holds for the conjugate momentum π_r found from the Lagrangian density $\mathcal{L}$ in (B2-2.2). That is, its appropriate value corresponding to the symbolism of (4) has the opposite sign of what Box 2-2 shows, i.e.,

$$\pi_r = -\frac{\partial \mathcal{L}}{\partial \dot{\phi}^r} \qquad \begin{cases}\text{(B2-2.2) re-written,} \\ \text{opposite sign of tensor analysis}\end{cases} \quad . \tag{5}$$

If we use the symbolism of (4) and (5), which parallels the rest of the book and other literature in QFT, then in (B2-2.3), reproduced below, the minus signs drop out and we get the same result.

$$\frac{\not{p}_i}{\pi_r} = \frac{\partial\mathcal{L}/\partial\dot{x}^i}{\partial\mathcal{L}/\partial\dot{\phi}^r} = \frac{\partial\dot{\phi}^r}{\partial\dot{x}^i} = \frac{\partial\phi^r/\partial t}{\partial x^i/\partial t} = \frac{\partial\phi^r}{\partial x^i} \quad \rightarrow \quad \not{p}_i = \pi_r\frac{\partial\phi^r}{\partial x^i} \quad \rightarrow \quad \not{p}^i = -\pi_r\frac{\partial\phi^r}{\partial x^i} \qquad \begin{cases}\text{(B2-2.3),} \\ \text{same result}\end{cases} \tag{6}$$

Perhaps now you can see why I fudged a bit in the book Box 2-2, and why I didn't want to get into this explanation there, at that time. It would have confused newcomers far more than enlightened them.