Aids to Chapter 2 of Student Friendly QFT

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Wholeness Chart 1. Comparing Different Sign Conventions for the Minkowski Metrics

	Common Relativity Text Convention	**Common QFT Text Convention**
Lorentz transformation	$x^{0'} = ct' = \frac{1}{\sqrt{1-v^2/c^2}}\underbrace{\left(ct - \frac{v}{c}x\right)}_{x^0 - \frac{v}{c}x^1}$ $x^{1'} = x' = \frac{1}{\sqrt{1-v^2/c^2}}\underbrace{(x - vt)}_{x^1 - \frac{v}{c}x^0}$ $x^{2'} = y' = y = x^2$ $x^{3'} = z' = z = x^3$	Same as at left
Lorentz transformation in matrix form	$\underbrace{\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & -\frac{v/c}{\sqrt{1-v^2/c^2}} & & \\ -\frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}}_{x^{\mu'} = \Lambda^{\mu}_{\nu} x^{\nu}}$	Same as at left
4D position vector	$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$ $x_\mu = (-ct, x, y, z) = (x_0, x_1, x_2, x_3) = (-x^0, x^1, x^2, x^3)$	$x^\mu = (ct, x, y, z) = (x^0, x^1, x^2, x^3)$ $x_\mu = (ct, -x, -y, -z) = (x_0, x_1, x_2, x_3) = (x^0, -x^1, -x^2, -x^3)$ Covariant form has opposite sign of at left.
Metric	$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\eta_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ negative of metric at left
Position vector length squared = interval squared	$(ds)^2 = dx^\mu dx_\mu = \left[dx^0 \;\; dx^1 \;\; dx^2 \;\; dx^3\right]\begin{bmatrix} -dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix}$ $= \left[dx^0 \;\; dx^1 \;\; dx^2 \;\; dx^3\right]\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \eta_{\mu\nu} dx^\mu dx^\nu$ $= -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ $= dx^0 dx_0 + dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3$ $= -(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2$	$(ds)^2 = dx^\mu dx_\mu = \left[dx^0 \;\; dx^1 \;\; dx^2 \;\; dx^3\right]\begin{bmatrix} dx^0 \\ -dx^1 \\ -dx^2 \\ -dx^3 \end{bmatrix}$ $= \left[dx^0 \;\; dx^1 \;\; dx^2 \;\; dx^3\right]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{bmatrix} = \eta_{\mu\nu} dx^\mu dx^\nu$ $= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ $= dx^0 dx_0 + dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3$ (same form as at left $= (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2$ (opposite sign from left)
Proper time τ on an object	$(cd\tau)^2 = -(ds)^2 = -\left(-(cdt)^2 + (dx)^2 + (dy)^2 + (dz)^2\right)$ for $dx = dy = dz = 0,\quad d\tau = d\tau$ (correct sign) Other notation: $(cd\tau)^2 = -(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ $= -dx^0 dx_0 - dx^1 dx_1 - dx^2 dx_2 - dx^3 dx_3$	$(cd\tau)^2 = (ds)^2 = \left((cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2\right)$ for $dx = dy = dz = 0,\quad d\tau = d\tau$ (correct sign) Other notation: $(cd\tau)^2 = (ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ $= dx^0 dx_0 + dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3$

4-vector	$w^\mu = (w^0, w^1, w^2, w^3)$ $w_\mu = (w_0, w_1, w_2, w_3) = (-w^0, w^1, w^2, w^3)$	$w^\mu = (w^0, w^1, w^2, w^3)$ $w_\mu = (w_0, w_1, w_2, w_3) = (w^0, -w^1, -w^2, -w^3)$ Covariant form has opposite sign of at left.
Magnitude of a 4-vector	$(w)^2 = w^\mu w_\mu = \eta_{\mu\nu} w^\mu w^\nu$ $= w^0 w_0 + w^1 w_1 + w^2 w_2 + w^3 w_3$ $= -w^0 w^0 + w^1 w^1 + w^2 w^2 + w^3 w^3$	$(w)^2 = w^\mu w_\mu = \eta_{\mu\nu} w^\mu w^\nu$ $= w^0 w_0 + w^1 w_1 + w^2 w_2 + w^3 w_3$ } same form as at left $= w^0 w^0 - w^1 w^1 - w^2 w^2 - w^3 w^3$ (opposite sign from left)
4-velocity	$u^\mu = \frac{dx^\mu}{d\tau} = \begin{bmatrix} u^0 \\ u^1 \\ u^2 \\ u^3 \end{bmatrix} = \gamma \begin{bmatrix} c \\ v^1 \\ v^2 \\ v^3 \end{bmatrix} = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \begin{bmatrix} c \\ v^1 \\ v^2 \\ v^3 \end{bmatrix}$ v^i = Newton velocity	Same as at left for contravariant form. Covariant form has opposite sign of at left.
4-velocity squared	$(u)^2 = u^\mu u_\mu = -c^2$ Massive particles.	$(u)^2 = u^\mu u_\mu = c^2$ (Just as $(ds)^2$ had opposite sign)
4-momentum	↓ Valid for all particles Massive particles $p^\mu = mu^\mu$. $p^\mu = \begin{bmatrix} p^0 \\ p^1 \\ p^2 \\ p^3 \end{bmatrix} = \begin{bmatrix} E/c \\ p^1 \\ p^2 \\ p^3 \end{bmatrix}$ $E = \frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$	Same as at left for contravariant form. Covariant form has opposite sign of at left.
4-momentum squared	$(p)^2 = p^\mu p_\mu = -m^2c^2$ Massive and massless particles. $p^\mu p_\mu = -\frac{E^2}{c^2} + p^i p^i = -m^2 c^2$	$(p)^2 = p^\mu p_\mu = m^2c^2$ Massive and massless particles. $p^\mu p_\mu = \frac{E^2}{c^2} - p^i p^i = m^2 c^2$
Invariance	Length of 4-vector invariant. (Same for all inertial observers.) Laws of nature have same form for all observers.	Same invariance as at left.
Example of invariant law	$F^\mu = m\frac{du^\mu}{d\tau}$ covariant form → $F_\mu = m\frac{du_\mu}{d\tau}$	Same as at left. $F^\mu = m\frac{du^\mu}{d\tau}$ covariant form → $F_\mu = m\frac{du_\mu}{d\tau}$
	<u>Spacetime Diagrams</u>	
Interval		
Inside light cone	Timelike $c\Delta t > \Delta x$ $(\Delta s)^2$ negative	Timelike $c\Delta t > \Delta x$ (same as at left) $(\Delta s)^2$ positive
Outside	Spacelike $c\Delta t < \Delta x$ $(\Delta s)^2$ positive	Spacelike $c\Delta t < \Delta x$ (same as at left) $(\Delta s)^2$ negative
On surface	Lightlike $c\Delta t = \Delta x$ $(\Delta s)^2 = 0$	Lightlike $c\Delta t = \Delta x$ (same as at left) $(\Delta s)^2 = 0$ (same as at left)

Note: String theory texts commonly use the relativity form of the metric instead of the QFT form.

Explicit Time Dependence

Consider the Hamiltonian in non-relativistic theory.

$$H\left(\dot{x}^i, x^i\right) = \tfrac{1}{2} m\left(\dot{x}^i\right)^2 + V\left(x^i\right) \qquad \text{e.g., } V\left(x^i\right) = -G\frac{mM}{r} = -G\frac{mM}{\left|x^i\right|} \text{ or } V\left(x^i\right) = mgx \tag{1}$$

There is no time dependence, explicitly in *H*, only dependence on x^i and its time derivative. Yet, as a particle moves in the potential field, the potential and kinetic energy of the particle changes in time. A comet moving towards the sun sees its potential energy become less (more negative) while its kinetic energy increase (it goes faster).

For example, the potential changes as the particle (comet) moves even though there is not explicit time dependence in *V*. This is because *x*, the solution to the problem (the particle motion as a function of time), is a function of time.

$$V\left(x^i\right) \qquad \text{where the problem solution } x^i = x^i(t)\,. \tag{2}$$

V is an implicit (indirect) function of *t*, but not an explicit function of *t*.

Now consider a planet in orbit around a star. Normally the potential *V* it feels is constant in time, since it is at a fixed radius and the star's gravity field is not changing. But suppose the star goes nova and blows off mass over time. That changes the potential *V* at the planet's orbital distance. (We imagine the planet stays in the same place during the nova, which in reality does not happen.)

But a solar physicist could model the change in potential of the star as it loses mass, as a function of time, i.e., find a mathematical form for *M*(*t*). Then, we would have

$$V\left(x^i, t\right), \text{ where } V \text{ is an } \underline{\text{explicit function of time}}. \tag{3}$$

What does this mean for derivatives? Well,

$$\frac{\partial V\left(x^i\right)}{\partial t} = 0 \qquad \frac{\partial V\left(x^i, t\right)}{\partial t} \neq 0\,. \tag{4}$$

How about total derivative of *V* with respect to time, which includes implicit plus explicit changes in *V* with time?

$$\frac{dV\left(x^i\right)}{dt} = \frac{\partial V\left(x^i\right)}{\partial x^i}\frac{dx^i}{dt} + \underbrace{\frac{\partial V\left(x^i\right)}{\partial t}}_{=0} = \frac{\partial V\left(x^i\right)}{\partial x^i}\frac{dx^i}{dt} \qquad \frac{dV\left(x^i,t\right)}{dt} = \frac{\partial V\left(x^i,t\right)}{\partial x^i}\frac{dx^i}{dt} + \underbrace{\frac{\partial V\left(x^i,t\right)}{\partial t}}_{\neq 0}. \tag{5}$$

Note that quantities like *H* are often expressed as functions of momentum and position, rather than velocity and position. Consider a Hamiltonian, for example, with time dependent potential,

$$H\left(p^i, x^i, t\right) = \tfrac{1}{2} m\left(p^i\right)^2 + V\left(x^i, t\right). \tag{6}$$

Then the total derivative would be

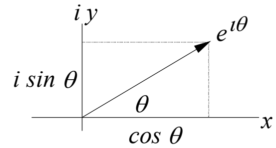
$$\frac{dH\left(p^i, x^i, t\right)}{dt} = \frac{\partial H}{\partial p^i}\frac{dp^i}{dt} + \frac{\partial H}{\partial x^i}\frac{dx^i}{dt} + \underbrace{\frac{\partial H}{\partial t}}_{\substack{\neq 0 \text{ in} \\ \text{this case}}}\,. \tag{7}$$

We have looked primarily at the simplest case of non-relativistic classical particle theory, but the same sort of total derivative relation of (7) is true for all parts of physics, including QFT.

Unitary Operators A Brief Look

Consider the complex number

$$e^{i\theta} = \cos\theta + i\sin\theta\ , \tag{8}$$

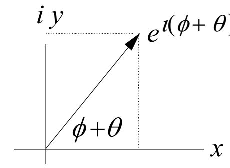


which represents a unit length vector in 2D complex space at an angle θ from the horizontal in the ccw direction. Its length squared, like any complex number, equals the number multiplied by its complex conjugate.

$$l^2 = \left(e^{i\theta}\right)^{\dagger} e^{i\theta} = e^{-i\theta} e^{i\theta} = 1\ . \tag{9}$$

Now multiply (8) another complex number $e^{i\phi}$.

$$e^{i\phi} e^{i\theta} = e^{i(\phi+\theta)}\ , \tag{10}$$



which is a unit length vector at an angle of $\phi + \theta$. $e^{i\phi}$ has rotated our original vector by an angle ϕ. We can think of $e^{i\phi}$ as a rotation operator in complex space. Whatever the quantity multiplied by i in the exponent is the amount the original vector (complex number) is rotated by. The length of the original vector does not change.

For a complex vector of length A,

$$Ae^{i\theta}\ , \tag{11}$$

$$e^{i\phi} Ae^{i\theta} = Ae^{i(\phi+\theta)} \tag{12}$$

the vector is rotated ϕ degrees without changing its length A. To reverse the operation of (12), do the following.

$$e^{-i\phi} e^{i\phi} Ae^{i\theta} = Ae^{i\theta}\ . \tag{13}$$

$e^{-i\phi}$ is the inverse operator of $e^{i\phi}$**.** It is also its complex conjugate. That is the definition of a unitary operator.

$$U = e^{i\phi} \qquad U^{-1} = e^{-i\phi} = U^{\dagger} \qquad U^{-1}U = U^{\dagger}U = 1\ . \tag{14}$$

In general, a <u>unitary operator U</u>

1. has an inverse equal to its complex conjugate transpose, $U^{\dagger}U = 1$ ($U^{-1} = U^{\dagger}$),
2. which implies it leaves the length of the vector it operates on unchanged.

Unitary operators commonly take the form of e with an imaginary exponent. They are used all of the time in quantum theories, since, when operating on a state vector (wave function, ket, ψ), they leave the length of the vector unchanged. For example, in NRQM,

$$\text{probability} = \int \psi^{\dagger}\psi\, dV = 1 \qquad U\psi = \psi' \qquad \psi^{\dagger}U^{\dagger} = \psi'^{\dagger} \qquad U^{\dagger} \text{ operates to the left here}$$
$$\int \psi^{\dagger} \underbrace{U^{\dagger}U}_{=1} \psi\, dV = \int \psi'^{\dagger}\psi'\, dV = 1 \tag{15}$$

The "length" of the "vector" ψ is the square root of the probability. For the old vector ψ or the transformed vector ψ', the length is the same. After the transformation, just as before, it equals one.

It is common in quantum theories to transform states to different forms for various reasons, including to obtain a form that is easier to analyze. When this is done, the total probability remains unchanged, which is essential, as any total probability other than one is a problem.

Any transformation with a unitary operator, such as $e^{i\phi}$ in (10) for example, is called a <u>unitary transformation</u>. It obeys what is called <u>unitarity</u>.

In addition to simple numbers, the exponent can include other operators (like H, for example), such as, where $\mathcal{O}$ stand for "operator",

$$\mathcal{O} = e^{-itH} \qquad \text{often written as } e^{-iHt} \text{ (but } H \text{ doesn't act on } t\text{)}\ . \tag{16}$$

Now let's look at the solution to Problem 12 in *Solutions to Problems for Student Friendly Quantum Field Theory, Vol.* 1.

Time Derivatives in Quantum Theory

1 An Example of the Equation of Motion in the Schrödinger Picture

Let's look at the time derivative of the expectation value of the x-direction 3-momentum in NRQM in the S.P, where the s super and subscripts indicate we are working in the S.P.

$$\frac{d\overline{p}_1}{dt} = {}_S\langle\psi|-i\left[p_1^S, H\right]|\psi\rangle_S + {}_S\langle\psi|\underbrace{\frac{\partial p_1^S}{\partial t}}_{=0}|\psi\rangle_S \qquad p_1^S \neq p_1^S(t) \qquad \text{(2-32) [26] } \mathit{SFQFT}$$

Consider the operator form of the momentum and the Hamiltonian in natural units where $\hbar$=1,

$$p_1^S = i\frac{\partial}{\partial x^1} \qquad H = \frac{p^2}{2m} + V(x) = -\frac{1}{2m}\frac{\partial}{\partial x^1}\frac{\partial}{\partial x^1} + V(x) = -\frac{1}{2m}\frac{\partial^2}{\partial^2 x^1} + V(x)\,. \tag{17}$$

Insert (17) into (2-32).

$$\frac{d\overline{p}_1}{dt} = {}_S\langle\psi|-i\left[i\frac{\partial}{\partial x^1}, -\frac{1}{2m}\frac{\partial^2}{\partial^2 x^1} + V(x)\right]|\psi\rangle_S = {}_S\langle\psi|\left[\frac{\partial}{\partial x^1}, -\frac{1}{2m}\frac{\partial^2}{\partial^2 x^1} + V(x)\right]|\psi\rangle_S$$

$$= {}_S\langle\psi|-\left[\frac{\partial}{\partial x^1}, \frac{1}{2m}\frac{\partial^2}{\partial^2 x^1}\right]|\psi\rangle_S + {}_S\langle\psi|\left[\frac{\partial}{\partial x^1}, V(x)\right]|\psi\rangle_S$$

$$= {}_S\langle\psi|\underbrace{-\left(\frac{\partial}{\partial x^1}\frac{1}{2m}\frac{\partial^2}{\partial^2 x^1}\right)}_{-\frac{1}{2m}\frac{\partial^3}{\partial^3 x^1}}|\psi\rangle_S - {}_S\langle\psi|\underbrace{-\left(\frac{1}{2m}\frac{\partial^2}{\partial^2 x^1}\frac{\partial}{\partial x^1}\right)}_{-\frac{1}{2m}\frac{\partial^3}{\partial^3 x^1}\text{cancels left}}|\psi\rangle_S$$

$$+ {}_S\langle\psi|\left(\frac{\partial}{\partial x^1}V(x)\right)|\psi\rangle_S - {}_S\langle\psi|\left(V(x)\frac{\partial}{\partial x^1}\right)|\psi\rangle_S \tag{18}$$

$$\frac{d\overline{p}_1}{dt} = {}_S\langle\psi|-\left(\frac{\partial}{\partial x^1}V(x)\right)|\psi\rangle_S - {}_S\langle\psi|-\left(V(x)\frac{\partial}{\partial x^1}\right)|\psi\rangle_S$$

$$= -{}_S\langle\psi|\left(\frac{\partial V(x)}{\partial x^1}\right)|\psi\rangle_S + \underbrace{{}_S\langle\psi|\left(V(x)\frac{\partial}{\partial x^1}\right)|\psi\rangle_S - {}_S\langle\psi|\left(V(x)\frac{\partial}{\partial x^1}\right)|\psi\rangle_S}_{\text{cancel}} \tag{19}$$

$$= -{}_S\langle\psi|\frac{\partial V(x)}{\partial x^1}|\psi\rangle_S = -\overline{\frac{\partial V(x)}{\partial x^1}} \qquad \left(\text{expection value of } -\frac{\partial V(x)}{\partial x^1}\right)$$

$$\frac{d\overline{p}_1}{dt} = \overline{\frac{\partial V(x)}{\partial x^1}} \tag{20}$$

We need to raise the index on $\overline{p}_1$ to get what we would measure with Cartesian coordinates. That entails a change in sign, so

$$\frac{-d\,\overline{p}^1}{dt} = \overline{\frac{\partial V(x)}{\partial x^1}} \qquad \rightarrow \qquad \frac{d\,\overline{p}^1}{dt} = -\overline{\frac{\partial V(x)}{\partial x^1}} \tag{21}$$

The expectation value of the total time derivative of the momentum equals the expectation value of the force (the negative of the gradient of the potential energy). This is Newton's second law, for what we expect to measure.

Bottom line: The relation (2-32) above, where momentum does not depend explicitly on time, i.e.,

$$\frac{d\bar{p}_1}{dt} = {}_S\langle\psi| -i\left[p_1^S, H\right]|\psi\rangle_S \,, \tag{22}$$

gives us the law of nature for momentum, i.e., the equation of motion for a particle, of $F = ma$. The commutator with the Hamiltonian gave us this result for the total time derivative of the momentum.

The General Law for Any Operator

See Wholeness Chart 2-4 [28].

$$\begin{gathered} \frac{d\bar{\mathcal{O}}}{dt} = {}_S\langle\psi|\left(-i\left[\mathcal{O}^S, H\right] + \frac{\partial \mathcal{O}^S}{\partial t}\right)|\psi\rangle_S \\ \frac{d\mathcal{O}^S}{dt} = \frac{\partial \mathcal{O}^S}{\partial t} = 0 \quad \text{usually (and always for us in this course)} \end{gathered} \tag{23}$$

2 Same Example with No Potential Energy

Suppose $V = 0$ in the above example. Then from (21) and what we know from before, momentum does not change in time (its total derivative is zero). That makes sense. If there is no potential, there is no force, so the momentum is unchanged. It is conserved.

Now look at (2-32) again, or equivalently (23) where the operator is momentum in the x^1 direction. We have no $V(x)$ term in H, just the momentum term of (17), which has the partial derivatives with respect to x^1. So, all the terms in the commutator of (23) have derivatives to some order with respect to x^1. But such derivatives always commute with one another. See the middle of (18).

In that case the commutator is zero. That means the time derivative of the expectation value of the momentum on the LH of (23) is zero. Momentum is conserved is the commutator is zero. And it is when $V(x) = 0$, which is what we know occurs for momentum when there is no potential (no force). Viola!

We can generalize.

Bottom line: If an operator commutes with the Hamiltonian, it is conserved. (At least its value as measured in the classical world is. That is, its expectation value is conserved.)

3 How About Another Operator Like the Hamiltonian.

Let's look at (23) for the Hamiltonian, where we assume (as we will from now on) that there is no explicit time dependence, i.e., $H \neq H(t)$, so the last term in (23) is zero.

$$\frac{d\bar{H}}{dt} = {}_S\langle\psi|\left(-i\left[H, H\right]\right)|\psi\rangle_S = -i\,{}_S\langle\psi|\left(HH - HH\right)|\psi\rangle_S = 0\,. \tag{24}$$

The Hamiltonian (energy we would measure, at least) is conserved. Duh … We knew this. But see how the general relation (23) gives us this result. Anything commutes with itself, and that includes the Hamiltonian. And from (23) that means the Hamiltonian does not change in time. It is conserved.

4 The Schrödinger Picture

As noted, all that we have done here is NRQM and is a standard part of advanced material in such courses. This is all in the Schrödinger picture, as NRQM is invariably taught.

Things change a bit when we go to the Heisenberg picture, but you need to understand the S.P. and its expectation values and time derivatives, before going to the H.P.