Page 16 – *Derivation going from (2-18) to (2-25)* by Luc Longtin

We start from the totally general 2X2 complex matrix in the form of (2-18), to which we impose the conditions (2-19) for a *special unitary group*. With a view of writing the matrix in a form that will reduce to the identity matrix in the limit where the independent parameters vanish, we write:

$$M = \begin{bmatrix} 1+\delta_1 & \delta_3 \\ \delta_4 & 1+\delta_2 \end{bmatrix} \qquad \text{…with conditions } M^\dagger M = I \text{ and Det } M = 1$$

In this form, at this point, the δ_i's, $i = 1, 2, 3, 4$, are arbitrary complex parameters.

From the *special* condition, Det $M = 1$, we must have: $(1+\delta_1)(1+\delta_2) - \delta_3\delta_4 = 1$

From the *unitary* condition, $M^\dagger M = I$, we must thus have:

$$M^\dagger M = \begin{bmatrix} 1+\delta_1^* & \delta_4^* \\ \delta_3^* & 1+\delta_2^* \end{bmatrix}\begin{bmatrix} 1+\delta_1 & \delta_3 \\ \delta_4 & 1+\delta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Component	*Expression*	*Component*	*Expression*
1-1	$(1+\delta_1^*)(1+\delta_1) + \delta_4^*\delta_4 = 1$	1-2	$(1+\delta_1^*)\delta_3 + \delta_4^*(1+\delta_2) = 0$
2-1	$(1+\delta_1)\delta_3^* + \delta_4(1+\delta_2^*) = 0$	2-2	$(1+\delta_2^*)(1+\delta_2) + \delta_3^*\delta_3 = 1$

Multiplying the 1-1 component with δ_3: $(1+\delta_1)(1+\delta_1^*)\delta_3 + \delta_3\delta_4\delta_4^* = \delta_3$

Using the 1-2 component, we can write: $-(1+\delta_1)(1+\delta_2)\delta_4^* + \delta_3\delta_4\delta_4^* = \delta_3$

Using the *special* condition relation, $(1+\delta_1)(1+\delta_2) = 1 + \delta_3\delta_4$, we have:

$$-(1+\delta_1)(1+\delta_2)\delta_4^* + \delta_3\delta_4\delta_4^* = -(1+\delta_3\delta_4)\delta_4^* + \delta_3\delta_4\delta_4^* = -\delta_4^* = \delta_3$$

So: $-\delta_4^* = \delta_3$ or $\delta_4 = -\delta_3^*$

Substituting into the 1-2 component relation, we get: $(1+\delta_1^*)\delta_3 - (1+\delta_2)\delta_3 = 0$

So: $\delta_1^* = \delta_2$ or $\delta_1 = \delta_2^*$

From the *special* condition: $(1+\delta_1)(1+\delta_2) - \delta_3\delta_4 = (1+\delta_1)(1+\delta_1^*) + \delta_3\delta_3^* = 1$

Or: $\delta_1 + \delta_1^* + \delta_1\delta_1^* + \delta_3\delta_3^* = 0$

Therefore: $$M = \begin{bmatrix} 1+\delta_1 & \delta_3 \\ \delta_4 & 1+\delta_2 \end{bmatrix} = \begin{bmatrix} 1+\delta_1 & \delta_3 \\ -\delta_3^* & 1+\delta_1^* \end{bmatrix}$$

NOTE: This matrix has *four* (real) parameters; namely the real and imaginary parts of δ_1 and δ_3. However, there is the *special* condition that imposes Det $M = 1$, so that there are, in fact, *three independent* (real) parameters. Expressing δ_1 and δ_3 in terms of *real* variables, we can write:

$\delta_3 = \alpha_2 + i\alpha_1$ so $\delta_3^* = \alpha_2 - i\alpha_1$

$$\delta_1 = \alpha_0 + i\alpha_3 \qquad \text{so} \qquad \delta_1^* = \alpha_0 - i\alpha_3$$

NOTE: In the above relations, *all* of the α parameters are *real*.

So: $$M = \begin{bmatrix} 1 + \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & 1 + \alpha_0 - i\alpha_3 \end{bmatrix} \qquad \text{with} \qquad \text{Det}\, M = 1$$

There remains to express the *special* condition in terms of the α_i's.

Thus: $$\delta_1 + \delta_1^* + \delta_1\delta_1^* + \delta_3\delta_3^* = 0 \qquad \text{...becomes...}$$

$$\alpha_0 + i\alpha_3 + \alpha_0 - i\alpha_3 + (\alpha_0 + i\alpha_3)(\alpha_0 - i\alpha_3) + (\alpha_2 + i\alpha_1)(\alpha_2 - i\alpha_1) = 0$$

$$2\alpha_0 + \alpha_0^2 + \alpha_3^2 + \alpha_2^2 + \alpha_1^2 = 0 \qquad \text{or} \qquad \alpha_0^2 + 2\alpha_0 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 0$$

So: $$\alpha_0 = -1 \pm \sqrt{1 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)} \qquad \text{or} \qquad \alpha_0 = \sqrt{1 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)} - 1$$

We choose the (+) sign solution, since we want α_0 to also vanish in the limit where the three independent α_i's vanish. We also note that since all the α's are *real*, the sum inside the square root cannot exceed unity; that is: $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 \leq 1$

NOTE: Of course, as planned from the beginning, in the limit where all the parameters tend to zero, the matrix M approaches the identity matrix, as desired.

Finally, we can also write in a form that will be useful later on, by factoring out a factor of i.

So: $$M = I + i\begin{bmatrix} -i\alpha_0 + \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -i\alpha_0 - \alpha_3 \end{bmatrix} \qquad \text{...where } \alpha_0 = \sqrt{1 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)} - 1$$